Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4295 Stastistical inference

Saturday 17 December 2011 9:00–13:00

Solutions

(Corrected 20 December 2011)

Problem 1

a) The likelihood function is given by $L(\lambda) = \lambda^{\sum x_i} e^{-n\lambda} / \prod x_i!$, $\ln L(\lambda) = \sum x_i \ln \lambda - n\lambda + \ln \prod x_i!$, $\partial \ln L(\lambda)/\partial\lambda = \sum x_i/\lambda - n$, $\partial^2 \ln L(\lambda)/\partial\lambda^2 = -\sum x_i/\lambda^2 < 0$, so that the MLE $\hat{\lambda}$ of λ is given by $\sum X_i/\hat{\lambda} - n = 0$, $\hat{\lambda} = \sum X_i/n = \bar{X}$, and by the invariance property the MLE of $e^{-\lambda}$ is $\hat{\theta} = e^{-\bar{X}}$.

b) $\ln f(x) = x \ln \lambda - \lambda - \ln x!$, $\partial \ln f(x)/\partial\lambda = x/\lambda - 1$, $\partial^2 f(x)/\partial\lambda^2 = -x/\lambda^2$, so the Cramér–Rao bound of an unbiased estimator of $e^{-\lambda}$ is $(de^{-\lambda}/d\lambda)^2/(-nE\partial^2 f(X_i)/\partial\lambda^2) = e^{-2\lambda}/(n\lambda/\lambda^2) = \lambda e^{-2\lambda}/n$.

c) $\sum U_i$ has the binomial $(n, e^{-\lambda})$ distribution, and by the central limit theorem $\sqrt{n}(\tilde{\theta} - e^{-\lambda}) \to N(0, e^{-\lambda}(1 - e^{-\lambda}))$ in distribution. By asymptotic efficiency of MLEs $\sqrt{n}(\hat{\theta} - e^{-\lambda}) \to N(0, \lambda e^{-2\lambda})$ (the same result would have been obtained by applying the central limit theorem to $\bar{X}$ and then the Delta method to $\hat{\theta} = e^{-\bar{X}}$). Then the ARE of $\tilde{\theta}$ with respect to $\hat{\theta}$ is $\lambda e^{-2\lambda}/(e^{-\lambda}(1 - e^{-\lambda})) = \lambda/(e^{\lambda} - 1)$. The preferable estimator is $\hat{\theta}$, since the ARE is near one only for λ near zero (the limit when $\lambda \to 0^+$ is 1 by L'Hôpital's rule) and always less than one ($e^{\lambda} - 1 > \lambda$ because $e^{\lambda} - 1 - \lambda$ is an increasing function for $\lambda > 0$) and has limit 0 when $\lambda \to \infty$.

d) $f(x) = 1/x! \cdot e^{-\lambda} \cdot e^{(\ln\lambda)x}$, showing that f is an exponential family and identifying $\sum X_i$ as a complete sufficient statistic for λ. Since U_1 is an unbiased estimator of $e^{-\lambda}$, $\theta^* = E(U_1 \mid \sum X_i)$ also has expected value $e^{-\lambda}$ and is a function of a complete sufficient statistic for λ, and is thus the UMVUE. To find that function explicitely, we first compute

$$E\left(U_1 \,\middle|\, \sum_{i=1}^n X_i = m\right) = P\left(U_1 = 1 \,\middle|\, \sum_{i=1}^n X_i = m\right) = \frac{P(U_1 = 1, \sum_{i=1}^n X_i = m)}{P(\sum_{i=1}^n X_i = m)}$$
$$= \frac{P(X_1 = 0, \sum_{i=2}^n X_i = m)}{P(\sum_{i=1}^n X_i = m)} = \frac{P(X_1 = 0)P(\sum_{i=2}^n X_i = m)}{P(\sum_{i=1}^n X_i = m)}$$
$$= \frac{e^{-\lambda} \cdot ((n-1)\lambda)^m e^{-(n-1)\lambda}/m!}{(n\lambda)^m e^{-n\lambda}/m!} = \left(\frac{n-1}{n}\right)^m = \left(1 - \frac{1}{n}\right)^m,$$

where we have used the fact that $\sum_{i=1}^n X_i$ and $\sum_{i=2}^n X_i$ have the Poisson distribution with parameter $n\lambda$ and $(n-1)\lambda$, respectively. So the UMVUE is $\theta^* = (1-1/n)^{\sum X_i} = (1-1/n)^{n\bar{X}}$. Note that $\lim_{n\to\infty}(1-1/n)^n = e^{-1}$, so that $\theta^* \to \hat{\theta}$ as $n \to \infty$.

e) Since $n\bar{X} = \sum X_i$ is Poisson distributed with parameter $n\lambda$, it has mgf given by $M(t) = Ee^{nt\bar{X}} = e^{n\lambda(e^t-1)}$. So $E\hat{\theta} = Ee^{-\bar{X}} = M(-1/n) = e^{-\lambda n(1-e^{-1/n})}$, and $E\hat{\theta}^2 = Ee^{-2\bar{X}} = M(-2/n) = e^{-\lambda n(1-e^{-2/n})}$, giving $\operatorname{Var}\hat{\theta} = E\hat{\theta}^2 - (E\hat{\theta})^2 = e^{-\lambda n(1-e^{-2/n})} - e^{-2\lambda n(1-e^{-1/n})}$. We already know that $E\theta^* = e^{-\lambda}$, and $E\theta^{*2} = E(1-1/n)^{2n\bar{X}} = Ee^{2n\bar{X}\ln(1-1/n)} = M(2\ln(1-1/n)) = e^{n\lambda((1-1/n)^2-1)} = e^{-\lambda(2-1/n)}$, so that $\operatorname{Var}\theta^* = E\theta^{*2} - (E\theta^*)^2 = e^{-2\lambda}(e^{\lambda}/n - 1)$. For $n = 20$ and $\lambda = 1$, $\operatorname{Var}\hat{\theta} = 0.006926$ and $\operatorname{Var}\theta^* = 0.006939$. Since θ^* is the unbiased estimator having lowest variance, and $\operatorname{Var}\hat{\theta} < \operatorname{Var}\theta^*$, $\hat{\theta}$ is not unbiased.

Problem 2

a) The cdf of $2\theta X_1^2$ is given by $P(2\theta X_1^2 \le y) = P\left(X_1 \le \sqrt{y/(2\theta)}\right)$, so the pdf of $2\theta X_1^2$ is given by

$$\frac{d}{dy}P(2\theta X_1^2 \le y) = \frac{d}{dy}P\left(X_1 \le \sqrt{\frac{y}{2\theta}}\right)$$
$$= f\left(\sqrt{\frac{y}{2\theta}}\right)\frac{d}{dy}\sqrt{\frac{y}{2\theta}} = 2\theta\sqrt{\frac{y}{2\theta}}e^{-y/2}\frac{1}{2\sqrt{y}}\frac{1}{\sqrt{2\theta}} = \frac{1}{2}e^{-y/2}$$

for $y > 0$, the pdf of the chi-squared distribution with 2 degrees of freedom.

b) Since each $2\theta X_i^2$ has the chi-squared distribution with 2 degrees of freedom and the X_i and hence the $2\theta X_i^2$ are independent, $2\theta \sum X_i^2$ has the chi-squared distribution with

$2n$ degrees of freedom. Then $1-\alpha = P\left(\chi^2_{2n,1-\alpha/2} \le 2\theta\sum X_i^2 \le \chi^2_{2n,\alpha/2}\right)$. Solving the inequalities with respect to θ yields

$$P\left(\frac{\chi^2_{2n,1-\alpha/2}}{2\sum X_i^2} \le \theta \le \frac{\chi^2_{2n,\alpha/2}}{2\sum X_i^2}\right) = 1-\alpha.$$

c) θ_0 not lying in the above confidence interval is indicative of H_1. Thus, rejecting H_0 when $\theta_0\sum X_i^2 < \frac{1}{2}\chi^2_{2n,1-\alpha/2} = 37.11$ or $\theta_0\sum X_i^2 > \frac{1}{2}\chi^2_{2n,\alpha/2} = 64.78$ has probability α if H_0 is true, and is a size α test.

d) Each $2\theta X_i^2$ has expected value 2 and standard deviation 2. By the central limit theorem, $\sqrt{n}(2\theta\sum X_i^2/n-2)/2 = \sqrt{n}(\theta\sum X_i^2/n-1) \to N(0,1)$ in distribution. So rejecting when $\sqrt{n}(\theta_0\sum X_i^2/n-1) < -z_{\alpha/2}$ or $\sqrt{n}(\theta_0\sum X_i^2/n-1) > z_{\alpha/2}$, that is, when $\theta_0\sum X_i^2 < n - z_{\alpha/2}\sqrt{n} = 36.14$ or $\theta_0\sum X_i^2 > n + z_{\alpha/2}\sqrt{n} = 63.86$, is an approximate size α test.

e) The likelihood function is given by $L(\theta) = 2^n\theta^n(\prod x_i)e^{-\theta\sum x_i^2}$, $\ln L(\theta) = n\ln 2 + n\ln\theta + \ln\prod x_i - \theta\sum x_i^2$, $\partial \ln L(\theta)/\partial\theta = n/\theta - \sum x_i^2$, so the likelihood is maximal at $n/\sum x_i^2$. Let $\lambda = L(\theta_0)/L(n/\sum X_i^2)$ be the LRT. Then $-2\ln\lambda = \ln L(\theta_0) - \ln L(n/\sum X_i^2) = -2(n\ln\theta_0 - \theta_0\sum X_i^2 - n\ln(n/\sum X_i^2) + n) = 2n(\ln n - \ln(\theta_0\sum X_i^2) - 1) + 2\theta_0\sum X_i^2 \to \chi^2_1$ in distribution if H_0 is true. To get an approximate size α test, H_0 should be rejected if $-2\ln\lambda > \chi^2_{1,\alpha}$, that is, if $\theta_0\sum X_i^2 - n\ln(\theta_0\sum X_i^2) > \frac{1}{2}\chi^2_{1,\alpha} - n(\ln n - 1)$. With the given numbers, this becomes $\theta_0\sum X_i^2 - 50\ln(\theta_0\sum X_i^2) > -143.7$. The function given by $g(t) = t - 50\ln t$ is decreasing for $0 < t < 50$ and increasing for $t > 50$, and the points where $g(t) = -143.7$ is given by the problem. We can conclude that H_0 is rejected if $\theta_0\sum X_i^2 < 37.39$ or $\theta_0\sum X_i^2 > 65.17$