Norwegian University of Science and Technology
Department of Mathematical Sciences

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Exam in TMA4295 STATISTICAL INFERENCE

Monday 10.December 2012
Time: 09.00 - 13:00

Support:

Tabeller og formler i Statistikk, Tapir
Matematisk formelsamling, K.Rottmann
NTNU Calculator (HP30S eller Citizen SR-270X)
Personal, hand written, yellow peep sheet - A5-format

Sensur: Wednesday 10.January 2013

Problem 1

Use $X \to f(x|\sigma^2) = \text{Norm}(0, \sigma^2)$, which entails that X has a Normal probability density function (pdf) with given parameter $\mu = 0$ and unknown parameter σ^2. Consider a random sample: $\mathbf{X} : X_1, X_2, \ldots, X_n$ iid $\text{Norm}(0, \sigma^2)$.

a) Develop an expression for the maximum likelihood estimator (MLE) for σ^2 - denote the estimator $\widehat{\sigma^2}$.

Demonstrate that the estimator $\widehat{\sigma^2}$ is Gamma distributed, and develop expressions for the distribution parameters - see Tabeller og Formler i Statistikk.

Use the result above to find expressions for $\text{E}(\widehat{\sigma^2})$ and $\text{Var}(\widehat{\sigma^2})$.

b) Use the Cramer-Rao Inequality to justify that $\widehat{\sigma^2}$ is a uniform minimum variance unbiased estimator (UMVUE) for σ^2.

c) Use a Bayesian framework with likelihood $[X|\sigma^2] \rightarrow f(x|\sigma^2) = \text{Norm}(0, \sigma^2)$ and prior pdf $\sigma^2 \rightarrow \pi(\sigma^2) = \text{InvGam}(\alpha, \beta)$, which entails that the prior pdf is Inverse-Gamma with given distribution parameters α and β - see Tabeller og Formler i Statistikk.

Develop an expression for the posterior pdf $[\sigma^2|\mathbf{X} = \mathbf{x}] \rightarrow \pi(\sigma^2|\mathbf{x})$.

The Normal-/Inverse-Gamma-distributions belong to a special class of distributions with certain Bayesian characteristics - what do we call this class and which characteristics does it have ?

d) Use a squared error loss function, $L(\sigma^2, \widetilde{\sigma^2})$, for estimator $\widetilde{\sigma^2}$ with respect to σ^2. Specify the Bayesian estimator for σ^2 which minimizes the associated risk function - denote the estimator $\widetilde{\sigma^2_B}$.

Develop an expression for $\text{E}(\widetilde{\sigma^2_B})$. Is the estimator $\widetilde{\sigma^2_B}$ an unbiased estimator for σ^2 ? Justify the answer.

What is required from an estimator for it to be a consistent estimator for σ^2 ? Does the estimator $\widetilde{\sigma^2_B}$ satisfy these requirements ? Justify the answer.

e) Let the parameter values in the prior pdf $\pi(\sigma^2)$ be $\alpha = 4$ and $\beta = \frac{1}{6}$, and let the number of observations be $n = 20$.

The two estimators will then be:

$$\begin{aligned}
\widehat{\sigma^2} &= \frac{1}{20}\Sigma_i X_i^2 \\
\widetilde{\sigma^2_B} &= \frac{1}{26}\Sigma_i X_i^2 + \frac{6}{13}
\end{aligned}$$

Compare the two estimators by using a mean square error (MSE) criterion. Interpret and discuss the results.

Problem 2

Use the pdf:

$$X \rightarrow f(x|\beta) = \begin{cases} \frac{1}{\beta^2} x \exp\{-\frac{x}{\beta}\} & x \geq 0 \\ 0 & \text{else} \end{cases}$$

with unknown distribution parameter $\beta > 0$. Note that $\mathrm{E}(X) = 2\beta$ and that $\mathrm{Var}(X) = 2\beta^2$. Consider a random sample: $\mathbf{X} : X_1, X_2, \ldots, X_n$ iid $f(x|\beta)$, with $n = 25$ and observations with $\Sigma_i x_i = 15.0$ and $\Sigma_i x_i^2 = 27.0$.

a) Use the Factorization theorem to identify one sufficient statistic for β based on the sample.

Develop an expression for the maximum likelihood estimator (MLE) for $\sigma^2 = \mathrm{Var}(X) = 2\beta^2$ - denote this estimator $\widehat{\sigma^2}$.

Compute the value of the estimate for σ^2 based on the observations.

b) Develope an expression for an approximate $(1 - \alpha)$-confidence interval for σ^2 by using the asymptotic distribution of $\widehat{\sigma^2}$.

Compute the value of the interval borders based on the observations, and $\alpha = 0.05$.

c) Consider now the unknown distribution parameter β. The following hypothesis is forwarded:

$$H_0 : \beta = \beta_0 \text{ versus } H_1 : \beta \neq \beta_0$$

Use a level α score test to determine a rejection interval for the hypothesis H_0.

Perform the test based on the observations when $\beta_0 = \frac{1}{5}$ and $\alpha = 0.05$. Shall the hypothesis H_0 be rejected ?