SUGGESTED SOLUTION

EXAM

TMA 4295 Statistical Inference

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Ref: H. Omre / 15.11.2012

Oppg. 1

$$X \to f(x|\sigma^2) = Norm(0,\sigma^2) = (2\pi)^{-1/2}(\sigma^2)^{-1/2}\exp\left\{-\frac{x^2}{\sigma^2}\right\}$$

$$-\infty < x < \infty$$

$$\sigma^2 \geq 0$$

$$\mathbb{X}: X_1, \dots, X_n \text{ iid } f(x|\sigma^2)$$

a)

$$L(\sigma^2|\mathbf{x}) = \prod_{i=1}^{n}(2\pi)^{-1/2}(\sigma^2)^{-1/2}\exp\left\{-\frac{x_i^2}{2\sigma^2}\right\}$$

$$= (2\pi)^{-n/2}(\sigma^2)^{-n/2}\exp\left\{-\frac{\sum_i x_i^2}{2\sigma^2}\right\}$$

$$\log L(\sigma^2|\mathbf{x}) = -\frac{n}{2}\log 2\pi - n/2 \log\sigma^2 - \frac{\sum_i x_i^2}{2\sigma^2}$$

$$\frac{\partial}{\partial\sigma^2}\log L(\sigma^2|\mathbf{x}) = -n/2\,\frac{1}{\sigma^2} - \frac{\sum x_i^2}{2}\frac{1}{(\sigma^2)^2}(-1) = 0$$

$$\Downarrow$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_i x_i^2$$

Estimator:

$$\hat{\sigma}^2 = \frac{1}{n}\sum_i X_i^2$$

Recall:

$$Y = \frac{n\hat{\sigma}^2}{\sigma^2} = \sum_i\left(\frac{X_i}{\sigma}\right)^2 \to f_Y(y) = Chi(n) = 2^{-n/2}\left[\Gamma\left(\frac{n}{2}\right)\right]^{-1} y^{n/2-1}\exp\left\{-\frac{y}{2}\right\}$$

$$y \geq 0$$

$$\frac{\partial y}{\partial\hat{\sigma}^2} = \frac{n}{\sigma^2}$$

Substitute:

$$\hat{\sigma}^2 \leadsto f_{\hat{\sigma}^2}(\hat{\sigma}^2) = f_Y\left(\frac{n\hat{\sigma}^2}{\sigma^2}\right)\left|\frac{n}{\sigma^2}\right|$$

$$= 2^{-n/2}\left[\Gamma\left(\frac{n}{2}\right)\right]^{-1}\left[\frac{n\hat{\sigma}^2}{\sigma^2}\right]^{n/2-1}\exp\left\{-\frac{n\hat{\sigma}^2}{2\sigma^2}\right\}\cdot\frac{n}{\sigma^2}$$

$$= \left[\frac{2\sigma^2}{n}\right]^{-\frac{n}{2}}\left[\Gamma\left(\frac{n}{2}\right)\right]^{-1}\left[\hat{\sigma}^2\right]^{n/2-1}\exp\left\{-\frac{\hat{\sigma}^2}{\left[\frac{2\sigma^2}{n}\right]}\right\}$$

$$= \mathrm{Gam}\left(\frac{n}{2}, \frac{2\sigma^2}{n}\right) \leftarrow \text{see Tables!}$$

$$E(\hat{\sigma}^2) = \frac{n}{2}\cdot\frac{2\sigma^2}{n} = \sigma^2 \quad \text{unbiased for } \sigma^2$$

$$\mathrm{Var}(\hat{\sigma}^2) = \frac{n}{2}\left[\frac{2\sigma^2}{n}\right]^2 = \frac{2(\sigma^2)^2}{n}$$

b) From Cramér-Rao:

$$\mathrm{Var}(W(\underline{X})) \geq \frac{\left[\frac{d}{d\sigma^2}E_{\sigma^2}(W(\underline{X}))\right]^2}{-n\,E_{\sigma^2}\left[\frac{\partial^2}{(\partial\sigma^2)^2}\log L(\sigma^2|\underline{X})\right]} \quad = 1 \text{ unbiased}$$

unbiased estimator for σ^2!

$$= \frac{1}{-n\,E_{\sigma^2}\left[\frac{1}{2(\sigma^2)^2} - \frac{X^2}{(\sigma^2)^3}\right]}$$

$$= \frac{2(\sigma^2)^2}{n}$$

Hence $\hat{\sigma}^2 = \frac{1}{n}\sum_i X_i^2$ - unbiased and variance equal lower bound for unbiased estimators

$\Longrightarrow$ $\hat{\sigma}^2$ is UMVUE for σ^2.

c) Likelihood:

$$[\mathbb{X} \mid \sigma^2] \rightsquigarrow f(x \mid \sigma^2) = (2\pi)^{-n/2} (\sigma^2)^{-n/2} \exp\left\{-\frac{\sum x_i^2}{2\sigma^2}\right\}$$

Prior:

$$\sigma^2 \rightsquigarrow \pi(\sigma^2) = \text{InvGam}(\alpha; \beta) = \beta^{-\alpha} [\Gamma(\alpha)]^{-1} (\sigma^2)^{-(\alpha+1)} \exp\left\{-\frac{1}{\beta\sigma^2}\right\}$$

$$; \sigma^2 \geq 0$$

Posterior:

$$[\sigma^2 \mid \mathbb{X} = x] \rightsquigarrow \pi(\sigma^2 \mid x) = \text{const} \times f(x \mid \sigma^2)\, \pi(\sigma^2)$$

$$= \text{const}' \times (\sigma^2)^{-n/2} \exp\left\{-\frac{\sum x_i^2}{2\sigma^2}\right\} (\sigma^2)^{-(\alpha+1)} \exp\left\{-\frac{1}{\beta\sigma^2}\right\}$$

$$= \text{const} \times (\sigma^2)^{-(\alpha+\frac{n}{2}+1)} \exp\left\{-\left(\frac{\sum x_i^2}{2} + \frac{1}{\beta}\right)\frac{1}{\sigma^2}\right\}$$

$$= \text{InvGam}\left(\alpha + \frac{n}{2}, \left[\frac{\sum x_i^2}{2} + \frac{1}{\beta}\right]^{-1}\right)$$

Normal/InvGam - pdfs is a family of pdfs that are conjugate wrt σ^2. Hence, Normal likelihood and InvGam prior — result in InvGam posterior.

d) Define quadratic loss function:

$$L(\sigma^2, \tilde{\sigma}^2) = (\sigma^2 - \tilde{\sigma}^2)^2 = [\sigma^2 - \tilde{\sigma}^2(\mathbb{X})]^2$$

The risk function is:

$$R_{\tilde{\sigma}^2}(\sigma^2) = E_{\sigma^2}\left(L(\sigma^2, \tilde{\sigma}^2(\mathbb{X}))\right)$$

$$= \int (\sigma^2 - \tilde{\sigma}^2(\mathbf{x}))^2 f(\mathbf{x} \mid \sigma^2)\, d\mathbf{x}$$

Bayes risk:

$$R_{\tilde{\sigma}^2} = \int R_{\tilde{\sigma}^2}(\sigma^2)\, \pi(\sigma^2)\, d\sigma^2$$

↙ posterior expectation

For squared error loss: $\tilde{\sigma}^2 = E(\sigma^2 \mid \mathbb{X} = x)$

From

$$[\sigma^2 \mid \mathbb{X} = \mathbf{x}] \to \mathrm{InvGam}\left(\alpha + \frac{n}{2}, \left[\frac{\sum x_i^2}{2} + \frac{1}{\beta}\right]^{-1}\right)$$

$$\Downarrow$$

$$E[\sigma^2 \mid \mathbb{X} = \mathbf{x}] = \left[\frac{\sum x_i^2}{2} + \frac{1}{\beta}\right]\left[\alpha - 1 + \frac{n}{2}\right]^{-1}$$

Hence estimator:

$$\tilde{\sigma}_B^2 = \left[\alpha - 1 + \frac{n}{2}\right]^{-1}\left[\frac{\sum X_i^2}{2} + \frac{1}{\beta}\right] = \left[\alpha - 1 + \frac{n}{2}\right]^{-1}\left[\frac{n\hat{\sigma}^2}{2} + \frac{1}{\beta}\right]$$

Expectation:

$$E(\tilde{\sigma}_B^2) = \left[\alpha - 1 + \frac{n}{2}\right]^{-1}\left[\frac{n E\hat{\sigma}^2}{2} + \frac{1}{\beta}\right] = \left[\alpha - 1 + \frac{n}{2}\right]^{-1}\left[\frac{n\sigma^2}{2} + \frac{1}{\beta}\right]$$

hence not unbiased – unless $\alpha \to 1$ & $\beta \to \infty$.

$$Var(\tilde{\sigma}_B^2) = \left[\alpha - 1 + \frac{n}{2}\right]^{-2} \frac{n^2}{4} Var\, \hat{\sigma}^2$$

$$= \left[\alpha - 1 + \frac{n}{2}\right]^{-2} \frac{n}{2} (\sigma^2)^2$$

Requirement for consistency:

$$\lim_{n\to\infty} E(\tilde{\sigma}^2) = \sigma^2$$

$$\lim_{n\to\infty} Var(\tilde{\sigma}^2) = 0$$

Note:

$$E(\tilde{\sigma}^2) = \left[\frac{\alpha-1}{n} + \frac{1}{2}\right]^{-1}\left[\frac{\sigma^2}{2} + \frac{1}{n\beta}\right] \xrightarrow{n\to\infty} \sigma^2$$

$$Var(\tilde{\sigma}^2) = \frac{1}{n}\left[\frac{\alpha-1}{n} + \frac{1}{2}\right]^{-2}\frac{1}{2}(\sigma^2)^2 \xrightarrow{n\to\infty} 0$$

hence $\tilde{\sigma}^2$ is a consistent estimator for σ^2.

e) $\alpha = 4$ & $\beta = 1/6$ — $n = 20$

Prior: $E(\sigma^2) = \frac{1}{\beta(\alpha-1)} = 2$

$Var(\sigma^2) = \frac{1}{\beta^2(\alpha-1)^2(\alpha-2)} = 2$

The estimators:

$$\hat{\sigma}^2 = \frac{1}{20}\sum X_i^2$$

$$\tilde{\sigma}_B^2 = \frac{1}{26}\sum X_i^2 + \frac{6}{13} = \frac{10}{13}\hat{\sigma}^2 + \frac{6}{13}$$

The $MSE(\cdot)$ are:

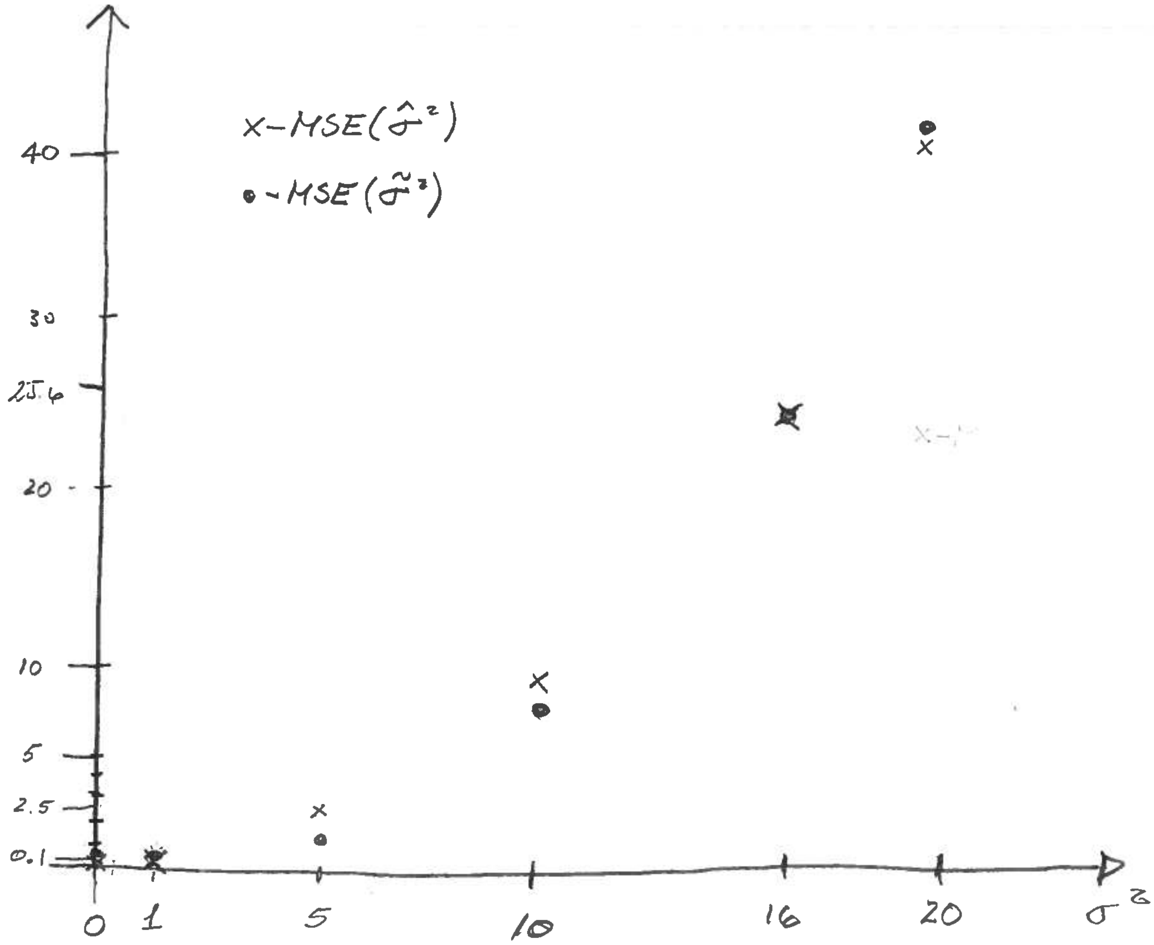
$$MSE(\hat{\sigma}^2) = Var(\hat{\sigma}^2) = \frac{2(\sigma^2)^2}{20} = \frac{(\sigma^2)^2}{10}$$

$$MSE(\tilde{\sigma}^2) = Var(\tilde{\sigma}^2) + \left[Bias(\tilde{\sigma}^2)\right]^2$$

$$= \left[\alpha-1+\frac{n}{2}\right]^{-2}\frac{n}{2}(\sigma^2)^2 + \left[\left[\alpha-1+\frac{n}{2}\right]^{-1}\left[(\alpha-1)\sigma^2 - \frac{1}{\beta}\right]\right]^2$$

$$= \left[\alpha-1+\frac{n}{2}\right]^{-2}\left[\left[(\alpha-1)^2+\frac{n}{2}\right](\sigma^2)^2 - \frac{2}{\beta}(\alpha-1)\sigma^2 + \frac{1}{\beta^2}\right]$$

$$= \frac{1}{169}\left[19(\sigma^2)^2 - 36\sigma^2 + 36\right]$$



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$MSE(\tilde{\sigma}^2) < MSE(\hat{\sigma}^2)$

NOTE: $\hat{\sigma}^2$ – unbiased – and uniform min variance

$\tilde{\sigma}^2$ – biased – and much smaller variance

– hence smaller MSE for some σ^2.

Oppg. 2

$$X \sim f(x|\beta) = \frac{1}{\beta^2} x \exp\left\{-\frac{x}{\beta}\right\} \qquad x \geq 0$$

with $\qquad\qquad\qquad\qquad \beta > 0$

$E(X) = 2\beta$

$Var(X) = 2\beta^2$

Consider:

$\underline{X}: X_1, \ldots, X_n$ iid $f(x|\beta)$

a)

$$f(\underline{x}|\beta) = \prod_{i=1}^{n} \frac{1}{\beta^2} x_i \exp\left\{-\frac{x_i}{\beta}\right\}$$

$$= \beta^{-2n} \prod_i x_i \exp\left\{-\frac{\sum x_i}{\beta}\right\}$$

$$= \underbrace{\beta^{-2n} \exp\left\{-\frac{\sum x_i}{\beta}\right\}}_{g(t(\underline{x})|\beta)} \underbrace{\prod_i x_i}_{h(\underline{x})} \longrightarrow t(\underline{x}) = \sum x_i$$

Hence sufficient statistic for β is $\sum_i X_i$.

$$L(\beta|\underline{x}) = \beta^{-2n} \exp\left\{-\frac{\sum x_i}{\beta}\right\} \prod_i x_i$$

$$\log L(\beta|\underline{x}) = -2n \log\beta - \sum x_i \frac{1}{\beta} + \sum_i \log x_i$$

$$\frac{\partial}{\partial\beta} \log L(\beta|\underline{x}) = \frac{-2n}{\beta} - \sum x_i \frac{1}{\beta^2}(-1) = 0$$

$$\hat{\beta} = \frac{1}{2n} \sum x_i$$

MLE estimator: $\hat{\beta} = \frac{1}{2n} \sum_i X_i$

Due to invariance of MLE wrt transformations

$$\hat{\sigma}^2 = \hat{Var}(X) = 2\hat{\beta}^2 = \frac{1}{2n^2}\left(\sum_i X_i\right)^2$$

Note:

$$\hat{\beta} = \frac{1}{50}\sum_i x_i = 0.30$$

hence

$$\hat{\sigma}^2 = 2 \cdot (0.30)^2 = 0.18$$

b) Recall

$$\sigma^2 = h(\beta) = 2\beta^2$$

hence by Delta method:

$$\hat{\sigma}^2 = h(\hat{\beta}) = h(\beta) + \frac{\partial h(\beta)}{\partial \beta}(\hat{\beta} - \beta) + \cdots\cdots$$

Since $\hat{\sigma}^2$ is MLE:

$$\hat{\sigma}^2 \xrightarrow{n \to \infty} \sigma^2$$

$$Var(\hat{\sigma}^2) = \left[\frac{\partial h(\beta)}{\partial \beta}\right]^2 Var(\hat{\beta} - \beta) + \cdots$$

and

$$\hat{Var}(\hat{\sigma}^2) = \left[\frac{\partial h(\beta)}{\partial \beta}\right]^2\Bigg|_{\beta = \hat{\beta}} \left[\hat{I}_n(\beta)\right]^{-1}$$

$$= \left[4\hat{\beta}\right]^2 \left[-\frac{\partial^2}{\partial \beta^2} \log h(\beta | x)\Big|_{\beta = \hat{\beta}}\right]^{-1}$$

$$= \left[4\hat{\beta}\right]^2 \left[\left[\sum_i x_i \frac{2}{\beta^3} - \frac{2n}{\beta^2}\right]_{\beta = \hat{\beta} = \frac{1}{2n}\sum x_i}\right]^{-1}$$

$$= \left[4\hat{\beta}\right]^2 \left[\frac{2n}{\hat{\beta}^2}\right]^{-1}$$

$$= \frac{8\hat{\beta}^4}{n}$$

Asymptotic pdf

$$\frac{\hat{\sigma}^2 - \sigma^2}{\sqrt{\hat{Var}(\hat{\sigma}^2)}} = \frac{\hat{\sigma}^2 - \sigma^2}{\sqrt{\frac{8\hat{\beta}^4}{n}}} \xrightarrow{n \to \infty} Norm(0,1)$$

$(1-\alpha)$-confidence interval for σ^2:

$$\left[\hat{\sigma}^2 - Z_{\alpha/2} \frac{4\hat{\beta}^2}{\sqrt{2n}}, \hat{\sigma}^2 + Z_{\alpha/2} \frac{4\hat{\beta}^2}{\sqrt{2n}}\right]$$

with

$$\hat{\beta} = \frac{1}{2n} \sum x_i = 0.30$$

$$\hat{\sigma}^2 = 2\hat{\beta}^2 = \frac{1}{2n^2}\left(\sum x_i\right)^2 = 0.18$$

$$Z_{\alpha/2} = 1.96$$

0.95-confidence interval for σ^2:

$$\left[0.18 - \underbrace{1.96 \frac{4 \cdot (0.3)^2}{\sqrt{50}}}_{0.05}, 0.18 + 1.96 \frac{4(0.3)^2}{\sqrt{50}}\right]$$

$$[0.13 , 0.23]$$

c) Hypothesis

$H_0: \beta = \beta_0$ versus $H_1: \beta \neq \beta_0$

Use Score test:

$$S_\beta(\mathbb{X}) = \frac{\partial}{\partial \beta} \log L(\beta | \mathbb{X}) = \sum \mathbb{X}_i \frac{1}{\beta^2} - \frac{2n}{\beta} =$$

NOTE:

$$E_\beta(S_\beta(\mathbb{X})) = E_\beta(\sum \mathbb{X}_i) \frac{1}{\beta^2} - \frac{2n}{\beta}$$

$$= \frac{2n\beta}{\beta^2} - \frac{2n}{\beta} = 0$$

moreover:

$$Var_\beta(S_\beta(\mathbb{X})) = E_\beta[S^2_\beta(\mathbb{X})]$$

$$= E_\beta\left[\left(\frac{\partial}{\partial \beta} \log L(\beta | \mathbb{X})\right)^2\right]$$

$$= -E_\beta\left[\frac{\partial^2}{\partial \beta^2} \log L(\beta | \mathbb{X})\right]$$

$$= -E_\beta\left[\frac{\partial}{\partial \beta}\left[\sum \mathbb{X}_i \frac{1}{\beta^2} - \frac{2n}{\beta}\right]\right]$$

$$= -E_\beta\left[-2\sum \mathbb{X}_i \frac{1}{\beta^3} + \frac{2n}{\beta^2}\right]$$

$$= 2E_\beta(\sum \mathbb{X}_i) \frac{1}{\beta^3} - \frac{2n}{\beta^2}$$

$$= 2 \cdot 2n\beta \cdot \frac{1}{\beta^3} - \frac{2n}{\beta^2}$$

$$= \frac{2n}{\beta^2}$$

Asymptotic pdf:

$$\frac{S_{\beta_0}(\mathbb{X})}{\sqrt{Var_{\beta_0}(S_{\beta_0}(\mathbb{X}))}} = \frac{\sum \mathbb{X}_i \frac{1}{\beta_0^2} - \frac{2n}{\beta_0}}{\sqrt{\frac{2n}{\beta_0^2}}}$$

$$= \frac{\frac{1}{2n}\sum \mathbb{X}_i - \beta_0}{\frac{\beta_0^2}{2n}\frac{\sqrt{2n}}{\beta_0}}$$

$$= \frac{\hat{\beta} - \beta_0}{\beta_0/\sqrt{2n}} \xrightarrow[n\to\infty]{} Norm(0,1)$$

Rejection region, level α test, for H_0:

$$R_\alpha: \left\{ x \,\middle|\, \left|\frac{\hat{\beta}(x) - \beta_0}{\beta_0/\sqrt{2n}}\right| > z_{\alpha/2} \right\}$$

hence

$$R_\alpha: \left\{ x \,\middle|\, \beta_0 - z_{\alpha/2}\frac{\beta_0}{\sqrt{2n}} > \hat{\beta}(x) \text{ or } \beta_0 + z_{\alpha/2}\frac{\beta_0}{\sqrt{2n}} < \hat{\beta}(x) \right\}$$

For observations, $\beta_0 = 1/5$ and $\alpha = 0.05$:

$$z_{\alpha/2}\frac{\beta_0}{\sqrt{2n}} = 1.96\frac{1/5}{\sqrt{50}} = 0.055$$

$$\hat{\beta}(x) = \frac{1}{20}\sum x_i = 0.30$$

Forkast H_0 fordi:

$$0.2 + 0.055 = 0.255 < \hat{\beta}(x) = 0.30 \,!$$