NTNU

Department of Mathematical Sciences

Examination paper for **TMA4295 Statistical Inference**

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Tabeller og Formler i Statistikk, Tapir

Matematisk formelsamling, K.Rottmann

NTNU certified calculator

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Problem 1

Use

$$X \to f(x|p) = \text{Geom}(p) = (1-p)^{x-1}p \; ; \; x = 1, 2, \dots \; ,$$

which entails that X has a Geometric probability mass function (pmf) with unknown parameter $0 < p \leq 1$. The associated expectation is $\mu = \text{E}(X) = \frac{1}{p}$ and variance is $\sigma^2 = \text{Var}(X) = \frac{1-p}{p^2}$. Consider a random sample: $\mathbf{X}_n : X_1, X_2, \dots, X_n$ iid Geom(p).

a) Demonstrate that the pmf $f(x|p)$ is a member of the exponential family of pmfs.

Demonstrate that the statistic $\text{T}(\mathbf{X}_n) = \Sigma_{i=1}^n X_i$ is a complete, sufficient statistic for parameter p, by using the characteristics of exponential families of pmfs.

Demonstrate that the maximum likelihood estimator (MLE) for p is $\hat{p} = \frac{n}{\Sigma_{i=1}^n X_i} = [\bar{X}]^{-1}$.

b) Specify the MLE's for μ and σ^2, and denote these estimators $\hat{\mu}$ and $\widehat{\sigma^2}$, respectively. Justify the answers.

Are the MLE's $\hat{p}$, $\hat{\mu}$ and $\widehat{\sigma^2}$ unbiased estimators ?

Consider only one observation, $X \to \text{Geom}(p)$, and define the following estimator for parameter p:

$$\tilde{p} = I[X = 1] = \begin{cases} 1 & \text{if } X = 1 \\ 0 & \text{else} \end{cases}$$

Demonstrate that $\tilde{p}$ is an unbiased estimator for parameter p.

c) Develop the expression for an unbiased estimator for parameter p based on the full random sample $\mathbf{X}_n$, by using the Rao-Blackwell Theorem.

Is this unbiased estimator a uniform minimum variance unbiased estimator (UMVUE) ? Justify the answer.

d) Demonstrate that the MLE $\hat{\mu}$ is a UMVUE for μ, by using the Cramer-Rao Theorem.

e) Specify the asymptotic distributions of the estimators $\hat{p}$, $\hat{\mu}$ and $\widehat{\sigma^2}$. Justify the answers.

In practice, how can numerical values for the asymptotic variances for $\hat{p}$, $\hat{\mu}$ and $\widehat{\sigma^2}$ be obtained ?

f) Consider the hypothesis

$$H_0 : p = p_0 \text{ versus } H_1 : p \neq p_0$$

for a fixed value $0 < p_0 \leq 1$.

Specify the rejection region for a level α asymptotic likelihood ratio test (LRT). Justify the answer.

Specify the corresponding asymptotic $(1-\alpha)$ confidence region for parameter p. Justify the answer.

Problem 2

Use

$$X \to f(x|\lambda) = \text{Pois}(\lambda) = \frac{\lambda^x}{x!} e^{-\lambda} \; ; \; x = 0, 1, \dots \; ,$$

which entails that the random variable X has a Poisson probability mass function (pmf) with unknown parameter $\lambda > 0$. See 'Tabeller og Formler i Statistikk'.

Consider a sample $\mathbf{X}$ consisting of three independent random variables:

$$\begin{aligned} X_1 &\to f(x|\lambda) \\ X_2 &\to f(x|2\lambda) \\ X_3 &\to f(x|3\lambda) \end{aligned}$$

Note that the sample is collected with different parameters.

a) Identify a sufficient statistic for λ based on the sample $\mathbf{X}$, by using the Factorization Theorem.

Develop an expression for the maximum likelihood estimator (MLE) for λ, denote it $\hat{\lambda}$.

Is the estimator $\hat{\lambda}$ a uniform minimum variance unbiased estimator (UMVUE) for λ based on the sample $\mathbf{X}$? Justify the answer by using the Cramer-Rao Theorem.

The inference of parameter λ shall now be made in a Bayesian setting.

Assign a prior probability density function (pdf) to parameter λ:

$$\lambda \to \pi(\lambda|\alpha,\beta) = \text{Gamm}(\alpha,\beta) = [\Gamma(\alpha)]^{-1}\beta^{-\alpha}\lambda^{\alpha-1}e^{\frac{-\lambda}{\beta}} \ ; \ \lambda > 0 \ ,$$

which entails that the random variable λ has a prior Gamma pdf with known hyper-parameters $\alpha, \beta > 0$. See 'Tabeller og Formler i Statistikk'.

b) Consider the sample $\mathbf{X} = \mathbf{x}$. Develop expressions for the Bayes estimate with associated estimation variance:

$$\begin{aligned} \tilde{\lambda} &= \text{E}(\lambda|\mathbf{x}) \\ \sigma^2_{\tilde{\lambda}} &= \text{Var}(\lambda|\mathbf{x}) \end{aligned}$$

Discuss the prior pdf's and the above expressions for the following asymptotic choices of hyper-parameters:

$$\begin{aligned} &\alpha \to 0 \ , \ \beta \to \infty \quad \text{and } \alpha\beta = \mu \\ &\alpha \to \infty \ , \ \beta \to 0 \quad \text{and } \alpha\beta = \mu \end{aligned}$$

with constant $\mu > 0$. Comment on the answers.

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