~~Exe.~~
Problem 1.

a) The n throws are independent; in each throw is $p_i = Pr(\text{dice shows "}i\text{"})$.

From Tables we have

$$f(\underset{\sim}{x}|\theta) = \frac{n!}{x_1! \cdots x_6!} \cdot \left(\frac{1}{6}-\theta\right)^{x_1} \left(\frac{1}{6}\right)^{x_2+x_3+x_4+x_5} \cdot \left(\frac{1}{6}+\theta\right)^{x_6}$$

$$= h(\underset{\sim}{x}) \cdot e^{x_1 \ln(\frac{1}{6}-\theta) + x_6 \ln(\frac{1}{6}+\theta)}$$

for $\underset{\sim}{x} \in A$ where $A = \{(x_1, \cdots x_6) : x_i \geq 0, \Sigma x_i = n, x_i \text{ integer}\}$

Then $h(\underset{\sim}{x}) = \frac{n!}{x_1! \cdots x_6!} \left(\frac{1}{6}\right)^{x_2+x_3+x_4+x_5}$

$T(\underset{\sim}{X}) = (X_1, X_6)$ is sufficient by the Factorization criterion.

b) $T(\underset{\sim}{X})$ is minimal-sufficient if

$$\frac{f(\underset{\sim}{x}|\theta)}{f(\underset{\sim}{y}|\theta)} \text{ does not depend on } \theta \iff T(\underset{\sim}{x}) = T(\underset{\sim}{y}).$$

Here $\frac{f(\underset{\sim}{x}|\theta)}{f(\underset{\sim}{y}|\theta)} = \frac{h(\underset{\sim}{x})}{h(\underset{\sim}{y})} e^{(x_1-y_1)\ln(\frac{1}{6}-\theta) + (x_6-y_6)\ln(\frac{1}{6}+\theta)}$.

($\Leftarrow$) If $T(\underset{\sim}{x}) = T(\underset{\sim}{y})$, then this clearly does not depend on θ.

$\Leftrightarrow$ Assume this does not depend on θ.

Then $f(\theta) = (x_1 - y_1)\ln\left(\frac{1}{6} - \theta\right) + (x_6 - y_6)\ln\left(\frac{1}{6} + \theta\right)$

is constant in θ, so $f'(\theta) = 0$ for all θ.

Now $f'(\theta) = -\dfrac{x_1 - y_1}{\frac{1}{6} - \theta} + \dfrac{x_6 - y_6}{\frac{1}{6} + \theta}$

which is 0 if $\left(\frac{1}{6} + \theta\right)(x_1 - y_1) = \left(\frac{1}{6} - \theta\right)(x_6 - y_6)$

or $(*)$ $\theta(x_1 - y_1 + x_6 - y_6) = \frac{1}{6}(x_6 - y_6 - x_1 + y_1)$
for all θ

If $x_1 - y_1 + x_6 - y_6 \neq 0$, then we can solve for θ, so $(*)$ does not hold for all θ.

Thus $(x_1 - y_1) + (x_6 - y_6) = 0$.

But then $(x_6 - y_6) - (x_1 - y_1) = 0$

and it follows that $x_6 - y_6 = x_1 - y_1 = 0$.

c) If $x_6 = x_1 = 0$, then $f(\underset{\sim}{x} \mid \theta)$ is constant in θ, so MLE is not defined.

Otherwise:

$$\frac{\partial \log f(\underset{\sim}{x} \mid \theta)}{\partial \theta} = \boxed{-\frac{x_1}{\frac{1}{6} - \theta} + \frac{x_6}{\frac{1}{6} + \theta} = 0}$$

$$\frac{x_1}{\frac{1}{6}-\theta} = \frac{x_6}{\frac{1}{6}+\theta}$$

$$\frac{x_1}{6} + \theta x_1 = \frac{x_6}{6} - \theta x_6$$

$$\theta(x_1+x_6) = \frac{1}{6}(x_6-x_1)$$

so
$$\hat{\theta} = \frac{1}{6}\cdot\frac{x_6-x_1}{x_1+x_6} \quad \text{is MLE.}$$

d) $H_0: \theta=0$ vs. $H_1: \theta \neq 0$

$$\lambda(\underset{\sim}{x}) = \frac{f(\underset{\sim}{x}|0)}{f(\underset{\sim}{x}|\hat{\theta})} = \frac{e^{x_1 \ln\frac{1}{6} + x_6 \ln\frac{1}{6}}}{e^{x_1 \ln(\frac{1}{6}-\hat{\theta}) + x_6 \ln(\frac{1}{6}+\hat{\theta})}}$$

$$= e^{x_1[\ln\frac{1}{6} - \ln(\frac{1}{6}-\hat{\theta})] + x_6[\ln\frac{1}{6} - \ln(\frac{1}{6}+\hat{\theta})]}$$

$\cancel{-2\ln\lambda(\underset{\sim}{x}) = -2x_1}$

$$= e^{x_1\left[\ln\frac{\frac{1}{6}}{\frac{1}{6}-\hat{\theta}}\right] + x_6\left[\ln\frac{\frac{1}{6}}{\frac{1}{6}+\hat{\theta}}\right]}$$

$$= e^{x_1[-\ln(1-6\hat{\theta})] + x_6[-\ln(1+6\hat{\theta})]}$$

so

$$-2\ln\lambda(\underset{\sim}{x}) = 2x_1\ln(1-6\hat{\theta}) + 2x_6\ln(1+6\hat{\theta})$$

With data:

$$\hat{\theta} = \frac{1}{6}\left[\frac{12}{32}\right] = \frac{1}{16} = \underline{0.0625}$$

which gives

$$-2\ln\lambda(\underset{\sim}{x}) = 2\cdot 10\cdot \ln\left(1-\frac{6}{16}\right) + 2\cdot 22\cdot \ln\left(1+\frac{6}{16}\right)$$

$$= 4.61$$

which is > 3.84 $(= \chi^2_{0.05,1})$.

Konklusjonen er derfor å forkaste H_0, dvs påstå at terningen er fikset.

e) $$\frac{\partial \log f(\underset{\sim}{x}\mid\theta)}{\partial\theta} = -\frac{x_1}{\frac{1}{6}-\theta} + \frac{x_6}{\frac{1}{6}+\theta}$$

$$\frac{\partial^2 \log f(\underset{\sim}{x}\mid\theta)}{\partial\theta^2} = -\frac{x_1}{\left(\frac{1}{6}-\theta\right)^2} - \frac{x_6}{\left(\frac{1}{6}+\theta\right)^2}$$

so that

$$I(\theta) = -E\left(\frac{\partial^2 \log f(\underset{\sim}{x}\mid\theta)}{\partial\theta^2}\right) = \frac{n\left(\frac{1}{6}-\theta\right)}{\left(\frac{1}{6}-\theta\right)^2} + \frac{n\left(\frac{1}{6}+\theta\right)}{\left(\frac{1}{6}+\theta\right)^2}$$

$$= \frac{n}{\frac{1}{6}-\theta} + \frac{n}{\frac{1}{6}+\theta} = \frac{6n}{1-6\theta} + \frac{6n}{1+6\theta}$$

$$= \frac{12n}{(1-6\theta)(1+6\theta)} = \underline{\frac{12n}{1-36\theta^2}}$$

CR lower bound is hence

$$\frac{1-36\theta^2}{12n}.$$

Thus $\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N\left(0, \frac{1-36\theta^2}{12}\right)$

An approximate 95% confidence interval is ~~therefore~~

$$\hat{\theta} \pm 1.96 \cdot \sqrt{\frac{1-36\hat{\theta}^2}{12 \cdot 100}}$$

$$0.0625 \pm 1.96 \cdot \sqrt{\frac{1-36 \cdot 0.0625^2}{12 \cdot 100}}$$

$$0.0625 \pm 0.0525$$

i.e. $\underline{(0.01, 01150)}$

Consider an LRT for $H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$

Then $\lambda(\underset{\sim}{x}) = e^{x_1[\ln(\frac{1}{6}-\theta_0) - \ln(\frac{1}{6}-\hat{\theta})] + x_6[\ln(\frac{1}{6}+\theta_0) - \ln(\frac{1}{6}+\hat{\theta})]}$

The approximate acceptance region is now

$$-2\ln\lambda(\underset{\sim}{x}) \leq \chi^2_{0.05,\,1} = 3.84$$

The confidence set is hence all θ_0 for which $-2\ln\lambda(\underset{\sim}{x}) \leq 3.84$

~~CR lower bound is hence~~

$$\cancel{\frac{1-36\theta^2}{12n}}$$

~~Thus~~ $\cancel{\sqrt{n}(\hat\theta-\theta) \xrightarrow{d} N\left(0, \frac{1-36\theta^2}{12}\right)}$

f) $$\underline{\underline{E(\tilde\theta)}} = \frac{1}{2n}\left[E(X_6) - E(X_1)\right]$$

$$= \frac{1}{2n}\left[n\cdot\left(\tfrac{1}{6}+\theta\right) - n\left(\tfrac{1}{6}-\theta\right)\right] = \underline{\underline{\theta}}$$

~~g)~~ $$\mathrm{Var}(\tilde\theta) = \frac{1}{4n^2}\cdot\left[\mathrm{Var}\,X_6 + \mathrm{Var}\,X_1 - 2\,\mathrm{Cov}(X_1,X_6)\right]$$

$$= \frac{1}{4n^2}\left[n\left(\tfrac{1}{6}+\theta\right)\left(\tfrac{5}{6}-\theta\right) + n\left(\tfrac{1}{6}-\theta\right)\left(\tfrac{5}{6}+\theta\right) + 2n\left(\tfrac{1}{6}+\theta\right)\left(\tfrac{1}{6}-\theta\right)\right]$$

$$= \underline{\frac{1-12\theta^2}{12n}}$$

g) Since $\dfrac{1-12\theta^2}{12n} > \dfrac{1-36\theta^2}{12n}$

can we not use CR lower bound to decide whether $\tilde\theta$ is UMVUE.

$T(\underset{\sim}{X})$ is not complete. By Def. 6.2.21 we need to find a $g(T)$ with expected value 0 for all θ, but which is not $\equiv 0$.

It is clear that

$$E_\theta\left[X_1 + X_6\right] = n\left(\frac{1}{6}-\theta\right) + n\left(\frac{1}{6}+\theta\right)$$

$$= \frac{n}{3}$$

Thus

$$g(T) = X_1 + X_6 - \frac{n}{3} \quad (\text{not always} = 0)$$

is such that $E_\theta\, g(T) = 0$ for all θ.

Thus T is not complete. Thus we cannot conclude that $\tilde{\theta}$ is UMVU.

Problem 2

a) Prob. of failing on each day is p.
The events of failing/not failing are independent over days.

$$M_Y(t) = E(e^{tY})$$

$$= \sum_{y=1}^{\infty} e^{ty}(1-p)^{y-1}p$$

$$= pe^t \sum_{y=1}^{\infty}[e^t(1-p)]^{y-1}$$

$$= pe^t \sum_{y=0}^{\infty}[e^t(1-p)]^{y}$$

$$= pe^t \cdot \frac{1}{1-e^t(1-p)} \quad \text{when } e^t(1-p) < 1$$

i.e.

$t + \ln(1-p) < 0$

$$= \frac{pe^t}{1-(1-p)e^t}$$

i.e.

$\underline{t < -\ln(1-p)}$

$EY = M_Y'(0)$

$$M_Y'(t) = \frac{pe^t(1-(1-p)e^t) + pe^t(1-p)e^t}{(1-(1-p)e^t)^2}$$

Set $t=0$:

$$\underline{\underline{EY}} = M_Y'(0) = \frac{p \cdot p + p(1-p)}{p^2} = \frac{p}{p^2} = \underline{\underline{\frac{1}{p}}}$$

b) $M_{X_n}(t) = E(e^{tX_n}) = E(e^{\frac{t}{n}Y_n})$

$$= M_{Y_n}\left(\frac{t}{n}\right) = \frac{\frac{p}{n}e^{\frac{t}{n}}}{1-(1-\frac{p}{n})e^{\frac{t}{n}}}$$ for $\frac{t}{n} < -\ln(1-\frac{p}{n})$

i.e.

$t < -n\ln(1-\frac{p}{n})$

$= n\cdot(-\ln(1-\frac{p}{n}))$

$> p$ for all n

so

$M_{X_n}(t)$ exists for $\underline{t < p}$

Let $n \to \infty$. Introduce $z = \frac{1}{n}$, so $z \to 0$.

Then $$M_{X_n}(t) = \frac{pze^{tz}}{1-(1-pz)e^{tz}}$$

Both numerator and denominator tends to 0 as $z \to 0$

L'Hôpital: limit is limit of

$$\frac{pe^{tz} + pzte^{tz}}{pe^{tz} - (1-pz)te^{tz}} \to \frac{p}{p-t}$$

$= \frac{1}{1-\frac{t}{p}}$ which is expon. distr. with expected value $\frac{1}{p}$.