NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for **TMA4295 Statistical Inference**

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Problem 1 The Pareto Distribution

Consider the Pareto distribution which is frequently used in economy and earth sciences. The probability density function (pdf) is

$$X \to f(x|\alpha, \beta) = \text{Par}(\alpha, \beta) = \begin{cases} \beta\alpha^{\beta}x^{-(\beta+1)} & x \geq \alpha \\ 0 & x < \alpha \end{cases}$$

with model parameters $\alpha > 0$ and $\beta > 2$.

The associated cumulative distribution function (cdf) is,

$$X \to F(x|\alpha, \beta) = \text{CPar}(\alpha, \beta) = \begin{cases} 1 - \alpha^{\beta}x^{-\beta} & x \geq \alpha \\ 0 & x < \alpha \end{cases}$$

The expectation is $\mu = \text{E}(X) = [\beta - 1]^{-1}\beta\alpha$ and the variance $\sigma^2 = \text{Var}(X) = [\beta - 1]^{-2}[\beta - 2]^{-1}\beta\alpha^2$.

Consider a random sample: $\mathbf{X}_n : X_1, X_2, \ldots, X_n$ iid $\text{Par}(\alpha, \beta)$.

Assume initially, in Point a), b) and c), that both parameters (α, β) are unknown.

a) Use the Factorization Theorem to identify sufficient statistics for the parameters (α, β).

b) Develop expressions for the maximum likelihood estimators (MLE) $(\hat{\alpha}, \hat{\beta})$ for the parameters (α, β).

Specify expressions for the MLE's $(\hat{\mu}, \hat{\sigma^2})$ for expectation μ and variance σ^2. Justify the answer.

c) Consider the hypothesis

$$H_0 : \beta = \beta_0 \text{ versus } H_1 : \beta \neq \beta_0$$

for a fixed value $\beta_0 > 2$.

Develop an expression for the rejection region R_c for a likelihood ratio test for the hypothesis concerning parameter β.

Thereafter, in Point d), e) and f), assume that the parameter α is a known value $\alpha_0 > 0$ while the parameter $\beta > 2$ is unknown.

A sufficient statistic for parameter β is $\mathrm{T}_1(\mathbf{X}_n) = \Sigma_{i=1}^{n}\log X_i$.

d) Verify that the sufficient statistic $\mathrm{T}_1(\mathbf{X}_n)$ is a complete, sufficient statistic.

Specify the MLE β^* for parameter β. Justify the answer.

e) Is the estimator β^* a consistent estimator for parameter β ? Justify the answer.

Develop the expression for the asymptotic variance of the estimator β^*.

Is the estimator β^* asymptotically efficient for parameter β ? Justify the answer.

f) In this point only, phrase the parameter inference in a Bayesian setting and assign a Gamma prior pdf to parameter β ,

$$\beta \to f(\beta|\lambda, \kappa) = \mathrm{Gam}(\lambda, \kappa) = \begin{cases} \lambda^{\kappa}[(\kappa-1)!]^{-1}\beta^{\kappa-1}\exp\{-\lambda\beta\} & \beta \geq 0 \\ 0 & \beta < 0 \end{cases}$$

for known hyperparameters, real $\lambda > 0$ and integer $\kappa > 0$.

Demonstrate that the posterior pdf of β given the realization of the random sample $\mathbf{x}_n$ is a Gamma pdf, and develop expressions for the posterior hyperparameters.

Specify the expression for the Bayesian estimator $\beta^+ = \mathrm{E}(\beta|\mathbf{X}_n)$ for parameter β.

Hereafter, in Point g) and h), assume that the parameter β is a known value $\beta_0 > 2$ while the parameter $\alpha > 0$ is unknown.

A sufficient statistic for parameter α is $\mathrm{T}_2(\mathbf{X}_n) = \mathrm{Min}\{X_1, \ldots, X_n\} = X_{(1)}$.

g) Consider the first observation in the random sample $X_1 \to \mathrm{Par}(\alpha, \beta_0)$ and define the following estimator for parameter α,

$$\alpha^+ = \beta_0^{-1}[\beta_0 - 1]X_1.$$

Demonstrate that the estimator α^+ is unbiased for the parameter α.

Use the estimator α^+ and the Rao-Blackwell Theorem to develop an improved unbiased estimator $\tilde{\alpha}$ for parameter α with less-or-equal variance.

h) Consider the MLE for parameter α

$$\alpha^* = X_{(1)}.$$

Develop the expression for the pdf of the estimator α^* and use this pdf to demonstrate that α^* is a consistent estimator for parameter α.

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