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Examination paper for **TMA4295 Statistical Inference**

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Problem 1 The Buffon's Needle

In 1733, Georges-Louis Leclerc (1707-1788) - titled ' Comte de Buffon' - designed an experiment and defined an estimator for assessing the value of the physical constant π - known to us today as being 3.1415926535.. . The 'Compte' estimated π, but there existed no statistical theory at the time so he had no deep insight into his estimation procedure. An extension of the original experiment was also defined by the 'Compte' in 1777 - unfortunately his calculations contained an error - and the correct solution was presented by Pierre-Simon Laplace (1749-1827) in 1812. Through history since then, the 'Buffon's needle' problems have drawn a lot of attention from statisticians. We shall recalculate some of these results in this Problem set.

Consider the 'Buffon's needle' experiment outlined in Figure 1a. In the 2D plane there is a set of parallell lines distance $d > 0$ apart. A 'needle' of length $0 < l \leq d$ is thrown onto the 2D plane with uniformly random location and orientation. Observe the random variable X:

$$X \to f(x|p_0, p_1) = \begin{cases} 0 & \text{- no crossing needle and lines} \quad \text{with prob } p_0 \\ 1 & \text{- one crossing needle and lines} \quad \text{with prob } p_1 \end{cases}$$

where the probabilities (p_0, p_1) can be calculated from the geometry of the experiment:

$$p_0 = 1 - 2\frac{l}{d}\frac{1}{\pi} = 1 - 2r\frac{1}{\pi}$$
$$p_1 = 2\frac{l}{d}\frac{1}{\pi} = 2r\frac{1}{\pi}$$

with $r = l/d$. Hence the random experiment is a Bernoulli trial with parameter $p = p_1$.

Conduct the experiment n times and observe the random sample:

$$\mathbf{X}_n : X_1, \ldots, X_n \text{ iid } f(x|p_0, p_1)$$

Focus of the experiment is on estimation of the constant π, but firstly parameter $p = 2r/\pi$ is evaluated.

a) Develop an expression for a sufficient statistic for p based on $\mathbf{X}_n$.
Is this sufficient statistic also complete ? Justify your answer.

Demonstrate that the expression for the maximum likelihood (ML) estimator for p based on $\mathbf{X}_n$ is,

$$\hat{p}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

b) Is the ML estimator $\hat{p}_n$ unbiased for p ? Justify your answer.

Is the ML estimator $\hat{p}_n$ also the uniform minimum variance unbiased (UMVU) estimator for p ? Justify your answer.

Consider the experiment itself - how can it be designed to obtain the smallest variance of the estimator $\hat{p}_n$?

Focus now on the constant π.

c) Specify the expression for the ML estimator $\hat{\pi}_n$ for π based on $\mathbf{X}_n$. Justify your answer.

Develop the expression for the asymptotic variance for the estimator $\hat{\pi}_n$.

Is the estimator $\hat{\pi}_n$ an asymptotically efficient estimator for π ? Justify your answer.

d) Develop the expression for the rejection region for the asymptotic likelihood ratio (LR) α-level test for the hypothesis

$$\mathcal{H}_0 : \pi = \pi_0 = 3.14 \text{ versus } \mathcal{H}_1 : \pi \neq \pi_0$$

Specify the expression for the corresponding asymptotic LR $(1-\alpha)$-confidence region for π. Justify your answer.

The 'Compte' extended the experiment in 1777 as outlined in Figure 1b, and this experiment was later termed the 'Buffon-Laplace's needle' experiment. In the 2D plane there are two orthogonal sets of parallell lines, both sets distance $d > 0$ apart. A 'needle' of length $0 < l \leq d$ is thrown onto the 2D plane with uniformly random location and orientation. Observe the random variable X ,

$$X \to f(x|p_0, p_1, p_2) = \begin{cases} 0 & \text{- no crossing needle and lines} \quad \text{with prob } p_0 \\ 1 & \text{- one crossing needle and lines} \quad \text{with prob } p_1 \\ 2 & \text{- two crossing needle and lines} \quad \text{with prob } p_2 \end{cases}$$

where the probabilities (p_0, p_1, p_2) can be calculated from the geometry of the experiment,

$$p_0 = 1 - (4 - r)r\frac{1}{\pi}$$
$$p_1 = 2(2 - r)r\frac{1}{\pi}$$
$$p_2 = r^2\frac{1}{\pi}$$

with $r = l/d$.

Conduct the experiment n times and observe the random sample:

$$\mathbf{X}_n : X_1, \ldots, X_n \text{ iid } f(x|p_0, p_1, p_2)$$

Focus of the extended experiment is also on estimation of the constant π, but firstly $\theta = 1/\pi$ is evaluated and thereafter π is evaluated.

e) Demonstrate that the ML estimator $\tilde{\theta}_n$ for θ based on $\mathbf{X}_n$ is

$$\tilde{\theta}_n = \frac{1}{n(4 - r)r}[N_1 + N_2]$$

with $N_j = \Sigma_{i=1}^n I(X_i = j)$ for $j = 1, 2$ where the indicator function $I(A)$ is 1 if A is true and 0 else.

Demonstrate that this ML estimator $\tilde{\theta}_n$ is also the UMVU estimator for θ.

f) Specify the ML estimator $\tilde{\pi}_n$ for π. Jusify your answer.

Develop the expression for the asymptotic variance of the estimator $\tilde{\pi}_n$.

Develop the expression for the asymptotic relative efficiency (ARE) of $\tilde{\pi}_n$ with respect to $\hat{\pi}_n$ from Point c.

Insert $l = d$, ie $r = 1$, and $\pi = 3.14$ in the expression for the ARE and calculate the numerical value. Interpret the result.

Problem 2

Use

$$X \to f(x|\theta) = Uni[\theta - 1/2, \theta + 1/2]\ ;\ x \in \mathcal{R}\ ;\ \theta \in \mathcal{R}$$

which entails that the random variable X is uniformly distributed in the interval $[\theta - 1/2, \theta + 1/2]$ with unknown model parameter θ.

Consider the random sample,

$$\mathbf{X}_n : X_1, \ldots, X_n \text{ iid } f(x|\theta)$$

The corresponding increasingly ordered random sample is denoted $X_{(1)}, \ldots, X_{(n)}$.

Define a random variable $U \to Uni[0,1]$ with random sample $\mathbf{U}_n : U_1, \ldots, U_n$ with correspondingly increasingly ordered random sample $U_{(1)}, \ldots, U_{(n)}$. The associated pdfs are,

$$U_{(i)} \to f_{U_{(i)}}(u_i) = \frac{n!}{(i-1)!(n-i)!} u_i^{i-1}(1-u_i)^{n-i}\ ;\ 0 \le u_i \le 1$$

hence Beta distributed $Beta(i, n-i+1)$ (see 'Tabeller og Formler i Statistikk'),

$$[U_{(i)}, U_{(j)}] \to f_{U_{(i)}U_{(j)}}(u_i, u_j) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} u_i^{i-1}(u_j - u_i)^{j-1-i}(1-u_j)^{n-j}$$
$$;\ 0 \le u_i \le u_j \le 1$$

Define the estimator for θ,

$$\hat{\theta}_n = \frac{1}{2}[X_{(1)} + X_{(n)}]$$

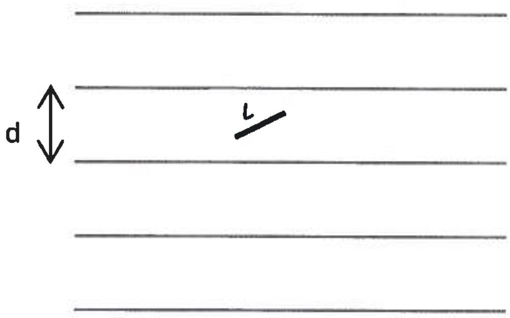
a) Develop the expressions for the sufficient statistics for θ based on the sample $\mathbf{X}_n$, by using the Factorization theorem.

Demonstrate that the estimator $\hat{\theta}_n$ is both a maximum likelihood (ML) estimator and an unbiased estimator for θ.

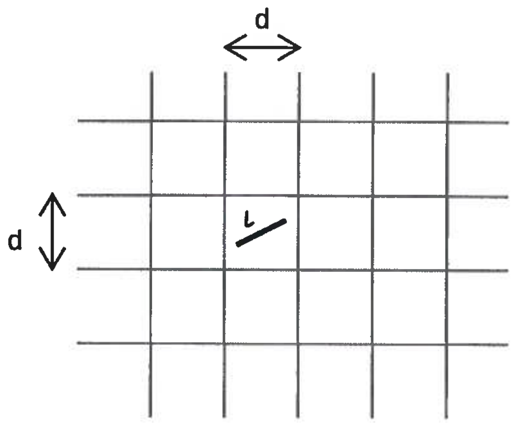
b) Develop the expression for the exact $(1-\alpha)$-confidence interval for θ based on the estimator $\hat{\theta}_n$.

Georges-Louis Leclerc, Comte de Buffon

Pierre-Simon Laplace



(a) Buffon's needle



(b) Buffon-Laplace's needle

Figure 1: Experimental design