→ SUGGESTED SOLUTION / H.Omre / 13.12.2016
Exam TMA4295 Statistical Inference
Nov 29. 2016

Problem 1 Buffon's Needle

$I(A) = \begin{cases} 1 & A \text{ true} \\ 0 & \text{else} \end{cases}$

$X \sim f(x|p_0,p_1) = p_1^{I(x=1)} p_0^{I(x=0)} = p^x(1-p)^{1-x} \quad x \in \{0,1\}$

$p_1, p_0, p \in [0,1]$

$p_0 = [1 - 2r\frac{1}{\pi}]$

$p = p_1 = 2r\frac{1}{\pi}$

Random sample:

$\mathbf{X}_n : X_1, \dots, X_n$ iid $f(x|p_0,p_1)$

$\mathbf{x}_n : x_1 \quad\quad x_n$

a) Consider

$$f(x|p_0,p_1) = f(x|p) = p^x(1-p)^{1-x} = (1-p)\left[\frac{p}{1-p}\right]^x$$

$$= \underbrace{(1-p)}_{c(p)} \exp\{\underbrace{[\ln p - \ln(1-p)]}_{\omega(p)} \underbrace{x}_{t(x)}\}$$

hence in exponential family.

Then the sufficient statistic is Theo 6.2.10,

$$Y_n = \sum_{i=1}^n X_i$$

Note $\omega(p) \in [-\infty, \infty]$, open set, then Theo 6.2.25 says

- Y_n is complete, sufficient statistic.

Note also that

$$Y_n \sim Bin_n(p) = \binom{n}{y} p^y (1-p)^{n-y}$$

$$E[Y_n] = np$$

$$Var[Y_n] = np(1-p)$$

The likelihood function for p based on $\mathbb{x}_n$ is:

$$L(p|\mathbb{x}_n) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{n\bar{x}_i}(1-p)^{n(1-\bar{x}_n)}$$

$$; \bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\ln L(p|\mathbb{x}_n) = n\bar{x}_i \ln p + n(1-\bar{x}_n)\ln(1-p)$$

$$\frac{d \ln L(p|\mathbb{x}_n)}{dp} = \frac{n\bar{x}_i}{p} + \frac{n(1-\bar{x}_n)}{1-p}(-1) = 0$$

$$(1-p)n\bar{x}_n = np - np\bar{x}_n \qquad \frac{d^2 \ln L}{dp^2} < 0$$

$$p = \bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$$

The ML estimator for p based on $\mathbb{X}_n$ is:

$$\hat{p}_n = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}_n = \frac{1}{n} Y_n$$

b) Unbiasedness from $Y_n \rightsquigarrow Bin_n(p)$

$$E[\hat{p}_n] = \frac{1}{n}E[Y_n] = \frac{1}{n}np = p$$

Since Y_n is complete, sufficient statistic for p - and $\hat{p}_n$ is unbiased for p - then from Theo 7.3.23 / Theo 7.5.1

$\hat{p}_n$ is UMVU estimator for p.

Note:

$$Var[\hat{p}_n] = \frac{1}{n^2}Var[Y_n] = \frac{1}{n^2}np(1-p) = \frac{p(1-p)}{n}$$

recall that $p = 2r/\pi$

$$Var[\hat{p}_n] = \frac{2r(\pi - 2r)}{n\pi^2} \quad ; \quad 0 < r = \ell/d \le 1$$

NOTE: Last question: Consider the experiment itself ....
- obviously unclerly phrased - SORRY!
Most students misunderstood !!
The answers do not count on the final grade !!

c) Recall that

$$\pi = 2r\frac{1}{p}$$

Hence from Theo 7.2.10 the ML estimator for π is:

$$\hat{\pi}_n = 2r\frac{1}{\hat{p}_n} = 2rn\left[\sum_{i=1}^{n} X_i\right]^{-1} = 2r\left[\bar{X}_n\right]^{-1}$$

Since $\hat{\pi}_n$ is the ML estimator for π; Theo 10.1.6:

$$E_\pi(\hat{\pi}_n) \xrightarrow[n\to\infty]{} \pi \qquad \text{consistent estimator}$$

$$Var_\pi(\hat{\pi}_n) \xrightarrow[n\to\infty]{} 0$$

moreover $\hat{\pi}_n$ is an asymptotic efficient estimator for π – from Theo 10.1.12

$$\sqrt{n}\left[\hat{\pi}_n - \pi\right] \xrightarrow[n\to\infty]{dist} Norm\left(0, v_{\hat{\pi}_n}(p)\right)$$

with

$$v_{\hat{\pi}_n}(p) = \left[\frac{d}{dp}2r\frac{1}{p}\right]^2 v_{\hat{p}_n}(p) = 4r^2\frac{-1}{p^4}v_{\hat{p}_n}(p) = \frac{\pi^4}{4r^2}v_{\hat{p}_n}(p)$$

where

$$v_{\hat{p}_n}(p) = \left[-E_p\left[\frac{d^2}{dp^2}\log f(X/p)\right]\right]^{-1}$$

$$= \left[-E_p\left[\frac{d^2}{dp^2}X\ln p - (1-X)\ln(1-p)\right]\right]^{-1}$$

$$= \left[-E_p\left[-X\frac{1}{p^2} - (1-X)\frac{1}{(1-p)^2}\right]\right]^{-1}$$

$$= \left[\frac{1}{p} + \frac{1}{1-p}\right]^{-1} = \left[\frac{1}{p(1-p)}\right]^{-1}$$

$$= p(1-p) = 2r\frac{1}{\pi}\left(1 - 2r\frac{1}{\pi}\right) = 2r\frac{1}{\pi^2}(\pi - 2r)$$

hence

$$v_{\hat{\pi}_n}(p) = \frac{\pi^4}{4r^2}\frac{2r}{\pi^2}(\pi - 2r) = \frac{\pi^2}{2r}(\pi - 2r)$$

Hence

$$\sqrt{n}\left[\hat{\pi}_n - \pi\right] \xrightarrow[n\to\infty]{dist} Norm\left(0, \frac{\pi^2}{2r}(\pi - 2r)\right)$$

d) The hypothesis is

$$H_0 : \pi = \pi_0 = 3.14 \quad \text{versus} \quad H_1 : \pi \neq \pi_0$$

The likelihood function for π given $\mathbb{x}_n$ is:

$$\mathcal{L}(\pi \mid \mathbb{x}_n) = \left[2r\tfrac{1}{\pi}\right]^{n\bar{x}_n}\left[1-2r\tfrac{1}{\pi}\right]^{n(1-\bar{x}_n)} \qquad \bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$= \left[1-2r\tfrac{1}{\pi}\right]^{n}\left[\frac{2r\frac{1}{\pi}}{1-2r\frac{1}{\pi}}\right]^{n\bar{x}_n}$$

The likelihood ratio (LR) test statistic is:

$$\underset{\in[0,1]}{\lambda(\mathbb{x}_n)} = \frac{\mathcal{L}(\overset{\downarrow H_0!}{\pi_0} \mid \mathbb{x}_n)}{\mathcal{L}(\underset{\uparrow \text{ML est!}}{\hat{\pi}_n} \mid \mathbb{x}_n)} = \left[\frac{1-2r\,{}^{1}\!/\!\pi_0}{1-\bar{x}_n}\right]^{n}\left[\frac{2r\frac{1}{\pi_0}}{\bar{x}_n}\right]^{n\bar{x}_n}\left[\frac{1-\bar{x}_n}{1-2r\,{}^{1}\!/\!\pi_0}\right]^{n\bar{x}_n}$$

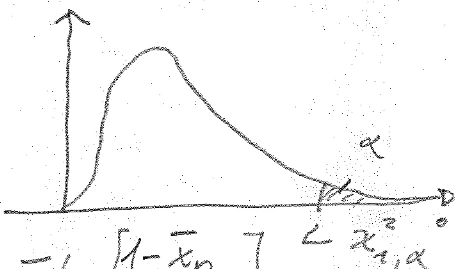
Asymptotic distribution of LR statistic Theo 10.3.1

$$\underset{\in[0,\infty)}{-2\log\lambda(\mathbb{x}_n)} \xrightarrow[n\to\infty]{\text{dist}} \chi^2_1$$

The rejection region for the asymptotic α-level test:

$$R_\alpha : \left\{ \mathbb{x}_n \,\middle|\, -2\left[n\log\left[\frac{1-2r\,{}^{1}\!/\!\pi_0}{1-\bar{x}_n}\right] + n\bar{x}_n\log\left[\frac{2r\,{}^{1}\!/\!\pi_0}{\bar{x}_n}\right] + n\bar{x}_n\log\left[\frac{1-\bar{x}_n}{1-2r\,{}^{1}\!/\!\pi_0}\right]\right] \geq \chi^2_{1,\alpha} \right\}$$

The corresponding asymptotic $(1-\alpha)$ confidence region by inversion of the acceptance region:



$$C(\mathbb{x}_n) : \left\{ \pi \,\middle|\, -2\left[n\log\left[\frac{1-2r\,{}^{1}\!/\!\pi}{1-\bar{x}_n}\right] + n\bar{x}_n\log\left[\frac{2r\,{}^{1}\!/\!\pi}{\bar{x}_n}\right] + n\bar{x}_n\log\left[\frac{1-\bar{x}_n}{1-2r\,{}^{1}\!/\!\pi}\right]\right] < \chi^2_{1,\alpha} \right\}$$

The 'Buffon-Laplace's needle' experiment

$$\mathbb{X} \leadsto f(x \mid p_0, p_1, p_2) = p_0^{I(x=0)}\, p_1^{I(x=1)}\, p_2^{I(x=2)} \qquad x \in \{0,1,2\}$$

$$p_0, p_1, p_2 \in [0,1]$$

$$p_0 = 1-(4-r)r\,\theta$$

$$p_1 = 2(2-r)r\,\theta \quad ; \quad \theta = 1/\pi$$

$$p_2 = r^2\theta$$

Random sample:

$\mathbb{X}_n : \mathbb{X}_1, \dots, \mathbb{X}_n$ iid $f(x \mid p_0, p_1, p_2)$

$\mathbb{x}_n : x_1, \dots, x_n$

e) Likelihood function

$$L(\theta \mid \mathbb{x}_n) = \prod_{i=1}^{n} f(x_i \mid \theta) = \prod_{i=1}^{n} p_0^{I(x_i=0)}\, p_1^{I(x_i=1)}\, p_2^{I(x_i=2)}$$

$$= [1-(4-r)r\,\theta]^{n_0} \cdot [2(2-r)r\,\theta]^{n_1} [r^2\theta]^{n_2}$$

$n_0 + n_1 + n_2 = n$

$n_j = \sum_{i=1}^{n} I(x_i = j)$; $j = 0,1,2$

$$\ln L(\theta \mid \mathbb{x}_n) = n_0 \ln[1-(4-r)r\,\theta] + n_1 \ln[2(2-r)r\,\theta] + n_2 \ln[r^2\theta]$$

ML estimator for θ:

$$\frac{d}{d\theta} \ln L(\theta \mid \mathbb{x}_n) = 0$$

$$\Downarrow$$

$$n_0 \frac{-(4-r)r}{1-(4-r)r\,\theta} + n_1 \frac{2(2-r)r}{2(2-r)r\,\theta} + n_2 \frac{r^2}{r^2\theta} = 0$$

$\Big\downarrow$ rearrange terms!

$n_0 = n - (n_1 + n_2)$

$$\tilde{\theta} = \frac{1}{(4-r)r} \frac{n_1 + n_2}{n}$$

hence

$$\tilde{\Theta}_n = \frac{1}{(4-r)r} \frac{N_1 + N_2}{n} \quad ; \quad N_0 = \sum_{i=1}^{n} I(\mathbb{X}_i = 0), \quad N_1 = \sum_{i=1}^{n} I(\mathbb{X}_i = 1), \quad N_2 = \sum_{i=1}^{n} I(\mathbb{X}_i = 2)$$

Define

$$N_{12} = N_1 + N_2$$

then

$$N_{12} \sim \mathrm{Bin}_n\big(\underbrace{(4-r)r\theta}_{p_1+p_2}\big) \qquad E[N_{12}] = n(4-r)r\theta$$

$$Var[N_{12}] = n(4-r)r\theta[1-(4-r)r\theta]$$

Unbiased:

$$E_\theta[\tilde{\theta}_n] = \frac{1}{(4-r)r\cdot n} E_\theta[N_{12}] = \theta$$

$$Var_\theta[\tilde{\theta}_n] = \frac{1}{(4-r)^2 r^2 n^2} Var_\theta[N_{12}] = \frac{[1-(4-r)r\theta]\theta}{(4-r)rn}$$

According to Cramér-Rao Theo 7.3.10

$$Var_\theta[W_n(\mathbb{X})] \geq \frac{1}{-E_\theta\left[\frac{d^2}{d\theta^2}\ln f(\mathbb{X}_n|\theta)\right]}$$

note

$$\frac{d^2}{d\theta^2}\ln [1-(4-r)r\theta]^{N_0}[2(2-r)r\theta]^{N_1}[r^2\theta]^{N_2}$$

$$\Big\downarrow \text{ some calculations}$$

$$= \frac{d}{d\theta}\left[N_0\frac{-(4-r)r}{1-(4-r)r\theta} + N_1\frac{1}{\theta} + N_2\frac{1}{\theta}\right]$$

$$= N_0\frac{-(4-r)^2r^2}{[1-(4-r)r\theta]^2} - N_1\frac{1}{\theta^2} - N_2\frac{1}{\theta^2}$$

$$-E_\theta[\cdot] = -\left[\frac{-n(4-r)^2r^2}{1-(4-r)r\theta} - \frac{n\,2(2-r)r}{\theta} - \frac{nr^2}{\theta}\right]$$

$E[N_j] = np_j \quad j=0,1,2$

$$= \frac{(4-r)rn}{[1-(4-r)r\theta]\theta}$$

hence

$$Var_\theta[W_n(\mathbb{X})] \geq \frac{[1-(4-r)r\theta]\theta}{(4-r)rn}$$

Hence

$$E_\theta[\tilde{\Theta}_n] = \Theta$$

$Var_\theta[\tilde{\Theta}_n]$ - hits C-R lower border

then $\tilde{\Theta}_n$ is the UMVU estimator for Θ.

f) Recall that

$$\pi = 1/\Theta$$

Hence from Theo 7.2.10 hence ML estimator

$$\tilde{\pi}_n = 1/\tilde{\Theta}_n = (4-r)rn\,[N_1+N_2]^{-1}$$

The ML-estimator is consistent and asymptotically efficient:

$$\sqrt{n}\,[\tilde{\pi}_n - \pi] \xrightarrow[n\to\infty]{dist} Norm(0, \sigma_{\tilde{\pi}_n}(\Theta))$$

with

$$\sigma_{\tilde{\pi}_n}(\Theta) = \left[\frac{d}{d\Theta}\frac{1}{\Theta}\right]^2 \sigma_{\tilde{\Theta}_n}(\Theta)$$

$$= \frac{1}{\Theta^4}\cdot\frac{[1-(4-r)r\Theta]\Theta}{(4-r)r}$$

$$\sigma_{\tilde{\pi}_n}(\pi)\Big|_{\Theta=\frac{1}{\pi}} = \pi^4\frac{[1-(4-r)r\frac{1}{\pi}]}{(4-r)r\pi}$$

$$= \frac{\pi^2}{(4-r)r}[\pi-(4-r)r]$$

hence

$$\sqrt{n}\,[\tilde{\pi}_n - \pi] \xrightarrow[n\to\infty]{dist} Norm\left(0, \frac{\pi^2}{(4-r)r}[\pi-(4-r)r]\right)$$

The asymptotic relative variance of $\tilde{\pi}_n$ with respect to $\hat{\pi}_n$ is

$$ARE(\tilde{\pi}_n, \hat{\pi}_n) = \frac{v_{\hat{\pi}_n}(\pi)}{v_{\tilde{\pi}_n}(\pi)} = \frac{\pi^2(\pi/2r - 1)}{\pi^2\left(\frac{\pi}{(4-r)r} - 1\right)}$$

$$= \frac{[\pi/2r - 1]}{[\pi/(4-r)r - 1]}$$

Insert for $l=d$, ie $r=1$ and $\pi = 3.14$,

$$ARE(\tilde{\pi}_n, \hat{\pi}_n) = \frac{.570}{.046} \approx \underline{\underline{12}}$$

The interpretion of this ARE is that:

- Each throw of the needle in the Buffon-Laplace experiment contains 12 times more information than in the Buffon experiment about π.
- If one throw 100 needles in the Buffon-Laplace experiment, one needs to throw 1200 in the Buffon experiment to obtain the same information about π.

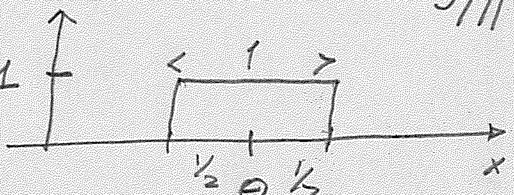
→ SURPRISINGLY large difference!

SEE ALSO:

Perlman & Wichura; 1975: 'Sharpening Buffon's Needle', The American Statistician, Vol. 29, No. 4, pp 157-163.

Problem 2.

$X \sim Uni[\theta - \frac{1}{2}, \theta + \frac{1}{2}]$



Note pivot:

$U = X - \theta + \frac{1}{2} \sim Uni[0, 1]$

hence:

$U_{(i)} = X_{(i)} - \theta + \frac{1}{2} \sim Beta(i, n-i+1)$

$E[U_{(i)}] = \frac{i}{n+1}$

$Var[U_{(i)}] = \dots$

consider:

$\hat{\theta}_n = \frac{1}{2}\left[X_{(1)} + X_{(n)}\right]$

a) Likelihood for θ based on X_n:

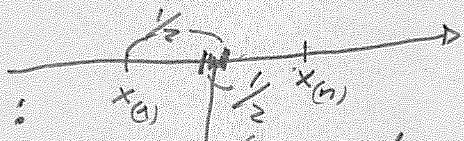
$I(A) = \begin{cases} 1 & A \text{ true} \\ 0 & \text{else} \end{cases}$

$$L(\theta | x_n) = \prod_{i=1}^{n} I\left(\theta - \tfrac{1}{2} \le x_i < \theta + \tfrac{1}{2}\right)$$

$$= \underbrace{I\left(\theta - \tfrac{1}{2} < x_{(1)}\right) \cdot I\left(x_{(n)} < \theta + \tfrac{1}{2}\right)}_{g(T(X_n)|\theta)}$$

Sufficient statistics for θ: $[X_{(1)}, X_{(n)}]$

$$L(\theta | x_n) = \begin{cases} 1 & \text{if } \theta < x_{(1)} + \frac{1}{2} \text{ and } \theta > x_{(n)} - \frac{1}{2} \\ 0 & \text{else} \end{cases}$$



$L(\theta|x_n) = 1$

Note that $\hat{\theta}_n$ is ML estimator for θ:

$$L(\hat{\theta}_n | x_n) = I\left(\hat{\theta}_n - \tfrac{1}{2} < x_{(1)}\right) I\left(x_{(n)} < \hat{\theta}_n + \tfrac{1}{2}\right)$$

$$= I\left(\tfrac{1}{2}(x_{(1)} + x_{(n)}) - \tfrac{1}{2} < x_{(1)}\right) I\left(x_{(n)} < \tfrac{1}{2}(x_{(1)} + x_{(n)}) + \tfrac{1}{2}\right)$$

$$= I\left((x_{(n)} - x_{(1)}) < 1\right) I\left((x_{(n)} - x_{(1)}) < 1\right)$$

$$= \underline{1} \quad \text{ok}$$

Note that $\hat{\theta}_n$ is unbiased for θ:

$$E[\hat{\theta}_n] = \tfrac{1}{2}\left[E[U_{(1)} + \theta - \tfrac{1}{2}] + E[U_{(n)} + \theta - \tfrac{1}{2}]\right]$$

$$= \tfrac{1}{2}\left[E[U_{(1)}] + E[U_{(n)}] + 2\theta - 1\right]$$

$$= \tfrac{1}{2}\left[\tfrac{1}{n+1} + \tfrac{n}{n+1} + 2\theta - 1\right] = \underline{\theta}$$

b) $\hat{\Theta}_n = \frac{1}{2}\left[X_{(1)} + X_{(n)}\right] = \overbrace{\frac{1}{2}\left[U_{(1)} + U_{(n)}\right]}^{S} + \Theta - \frac{1}{2}$

Note:

$$f_{U_{(1)}U_{(n)}}(u_1, u_n) = n(n-1)\left[u_n - u_1\right]^{n-2} \quad ; \quad 0 < u_1 < u_2 < 1$$

consider s:

$$s = \frac{1}{2}[u_n + u_1] \qquad \Rightarrow \qquad u_1 = s - t \qquad\qquad J = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

$$t = \frac{1}{2}[u_n - u_1] \qquad\qquad u_n = s + t$$

Transformation:

$$f_{ST}(s,t) = f_{U_{(1)}U_{(n)}}(s-t, s+t)\,|J|$$

$$= n(n-1)\left[(s+t) - (s-t)\right]^{n-2} \cdot 2$$

$$= 2^{n-1} n(n-1) t^{n-1}$$

$0 < s < 1$

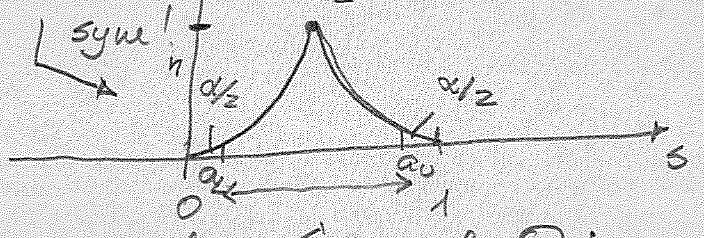
$0 < t < \min(s, 1-s)$

$$f_S(s) = \int f_{ST}(s,t)\,dt$$

$$= 2^{n-1} n(n-1) \int_0^{\min(s,1-s)} t^{n-2}\,dt$$

$$= \begin{cases} 2^{n-1} n\, s^{n-1} & 0 < s < \frac{1}{2} \\ 2^{n-1} n (1-s)^{n-1} & \frac{1}{2} < s < 1 \end{cases}$$

$S = \frac{1}{2}\left[U_{(1)} + U_{(n)}\right] \longrightarrow$



$(1-\alpha)$ confidence interval for S and Θ:

$$\text{Prob}\left\{a_L < \frac{1}{2}\left[U_{(1)} + U_{(n)}\right] < a_U\right\} \qquad a_U = 1 - a_L$$

$$\text{Prob}\left\{a_L + \Theta - \frac{1}{2} < \underbrace{\frac{1}{2}\left[U_{(1)} + U_{(n)}\right] + \Theta - \frac{1}{2}}_{\hat{\Theta}_n} < a_U + \Theta - \frac{1}{2}\right\}$$

$$\text{Prob}\left\{\hat{\Theta}_n - a_U + \frac{1}{2} < \Theta < \hat{\Theta}_n - a_L + \frac{1}{2}\right\}$$

$$\text{Prob}\left\{\hat{\Theta}_n + a_L - \frac{1}{2} < \Theta < \Theta_n - a_L + \frac{1}{2}\right\}$$

where

a_L is the $\alpha/2$-quantile of $f_S(s)$

$$\int_0^{a_L} 2^{n-1} n s^{n-1} ds = \alpha/2$$

$$\Downarrow$$

$$2^{n-1} \Big/_0^{a_L} s^n = \alpha/2$$

$$\Downarrow$$

$$2^n a_L^n = \alpha$$

$$\Downarrow$$

$$a_L = \frac{\alpha^{1/n}}{2}$$

The $(1-\alpha)$ confidence interval for θ is:

$$C(\mathbb{X}_n) = \left[\hat{\theta}_n + \frac{\alpha^{1/n}}{2} - \frac{1}{2} < \theta < \hat{\theta}_n - \frac{\alpha^{1/n}}{2} + \frac{1}{2} \right]$$

Note that

$$\alpha^{1/n} \xrightarrow[n\to\infty]{} 1 \Rightarrow C(\mathbb{X}_n) \xrightarrow[n\to\infty]{\epsilon_n \to 0} \left[\hat{\theta} - \epsilon_n < \theta < \hat{\theta}_n + \epsilon_n \right]$$