## Bayes and the normal distribution

Exercise 7.22

$X_1,\ldots,X_n \sim N\left(\theta,\sigma^2\right)$. $\pi(\theta) \sim N\left(\mu,\tau^2\right)$. $\mu,\tau^2$ and $\sigma^2$ are known.

Then $\bar{X} \sim N\left(\theta,\dfrac{\sigma^2}{n}\right)$

We get $f\left(\bar{x},\theta\right) = f\left(\bar{x}\middle|\theta\right)\pi\left(\theta\right) = \left(2\pi\right)^{-1} n^{\frac{1}{2}}\sigma^{-1}\tau^{-1}e^{-\frac{1}{2}\left\{\frac{\left(\sqrt{n}\left(\bar{x}-\theta\right)\right)^2}{\sigma^2}+\frac{(\theta-\mu)^2}{\tau^2}\right\}}$

Let us consider the exponent:

$$\frac{\left(\sqrt{n}\left(\bar{x}-\theta\right)\right)^2}{\sigma^2}+\frac{\left(\theta-\mu\right)^2}{\tau^2}=\frac{n\bar{x}^2-2n\bar{x}\theta+n\theta^2}{\sigma^2}+\frac{\theta^2-2\mu\theta+\mu^2}{\tau^2}$$

$$=\theta^2\left(\frac{1}{\tau^2}+\frac{n}{\sigma^2}\right)-2\theta\left(\frac{\mu}{\tau^2}+\frac{n\bar{x}}{\sigma^2}\right)+\left(\frac{n\bar{x}^2}{\sigma^2}+\frac{\mu^2}{\tau^2}\right)$$

$$=\left(\frac{\sigma^2+n\tau^2}{\sigma^2\tau^2}\right)\left[\theta^2-2\theta\left(\frac{\sigma^2\mu+n\bar{x}\tau^2}{\sigma^2+n\tau^2}\right)+\left(\frac{\sigma^2\mu+n\bar{x}\tau^2}{\sigma^2+n\tau^2}\right)^2-\left(\frac{\sigma^2\mu+n\bar{x}\tau^2}{\sigma^2+n\tau^2}\right)^2+\left(\frac{\tau^2n\bar{x}^2+\mu^2\sigma^2}{\sigma^2+n\tau^2}\right)\right]$$

$$=\left(\frac{\sigma^2+n\tau^2}{\sigma^2\tau^2}\right)\left[\theta^2-\left(\frac{\sigma^2\mu+n\bar{x}\tau^2}{\sigma^2+n\tau^2}\right)\right]^2-\frac{\left(\sigma^2\mu+n\bar{x}\tau^2\right)^2}{\left(\sigma^2+n\tau^2\right)\sigma^2\tau^2}+\left(\frac{\tau^2n\bar{x}^2+\mu^2\sigma^2}{\sigma^2\tau^2}\right)$$

$$=\left(\frac{\sigma^2+n\tau^2}{\sigma^2\tau^2}\right)\left[\theta^2-\left(\frac{\sigma^2\mu+n\bar{x}\tau^2}{\sigma^2+n\tau^2}\right)\right]^2-\frac{n\left(\bar{x}-\mu\right)^2}{\left(\sigma^2+n\tau^2\right)}$$

$$f\left(\bar{x},\theta\right)=\frac{\sqrt{\sigma^2+n\tau^2}}{\sqrt{2\pi}\sigma\tau}e^{\frac{-1}{2}\left\{\frac{\left[\theta^2-\left(\frac{\sigma^2\mu+n\bar{x}\tau^2}{\sigma^2+n\tau^2}\right)\right]^2}{\frac{\sigma^2\tau^2}{\sigma^2+n\tau^2}}\right\}}\cdot\frac{n^{\frac{1}{2}}}{\sqrt{2\pi}\sqrt{\sigma^2+n\tau^2}}e^{\frac{-1}{2}\left\{\frac{(\bar{x}-\mu)^2}{\frac{\left(\sigma^2+n\tau^2\right)}{n}}\right\}}$$

This is the product of two normal distributions and the first is $\pi\left(\theta\middle|\bar{x},\mu,\tau,\sigma^2\right)$ and the posterior mean

is: $E\left[\theta\middle|\bar{x}\right]=\dfrac{\mu\frac{\sigma^2}{n}}{\frac{\sigma^2}{n}+\tau^2}+\dfrac{\bar{x}\tau^2}{\frac{\sigma^2}{n}+\tau^2}$