Exercise 8. TMA4295

Problem 1

Let $X \sim \text{gamma}(\alpha, \beta)$ where $\alpha, \beta > 0$ (see the density on page 624 in book).

Show that

$$E(\ln X) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \ln \beta.$$

Hint: It can be shown that differentiation with respect to α is allowed under the integral sign in $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du$.

Problem 2

Let $X_1, \ldots, X_n$ be i.i.d. uniformly distributed on the interval $[0, \theta]$. It was shown in exercise 7 that the statistic

$$T(\boldsymbol{X}) = \max\{X_1, \ldots, X_n\}$$

is sufficient for θ.

a) Find the moment estimator of θ. Can this be written as a function of $T(\boldsymbol{X})$? Give a comment.

b) Let $n = 3$ and assume that the observations are 0.1, 0.9, 8.0. Compute the moment estimate. Is this estimate of θ reasonable?

c) Derive the MLE for θ. Find its expectation, variance and Mean Squared Error (MSE), i.e. $E(\hat{\theta} - \theta)^2)$.

d) Find an unbiased estimator for θ on the form const$\cdot T(\boldsymbol{X})$. Find the estimator's variance and compare with the MSE of, respectively, the moment estimator and the MLE.

Problem 3

Let $X_1, \ldots, X_n$ be i.i.d. $N(\mu, \mu^2)$, where μ is to be estimated. Discuss the use of the moment method, maximum likelihood method and the general result for MLE in exponential families (from the lectures and the note by Rue and Skaflestad).

Problem 4 (Bayes estimation)

Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be n observations from $N(\theta, \sigma^2)$, where σ^2 is known. Assume that we have given a prior distribution for θ given by $N(m, \tau^2)$.

a) Show that the posterior distribution for θ given $\boldsymbol{X} = \boldsymbol{x}$ is given by

$$N\left(\frac{\sigma^2}{\sigma^2 + n\tau^2} m + \frac{n\tau^2}{\sigma^2 + n\tau^2} \bar{x}, \frac{\sigma^2\tau^2}{\sigma^2 + n\tau^2}\right)$$

Try to minimize the computation (it may be cumbersome otherwise!) See also Example 7.2.16 in the book.

b) Which family of distributions is hence conjugate to the normal distribution?

c) What is the Bayes-estimator for θ? Which is the weight it puts on the prior knowledge versus the information from the data $\boldsymbol{X}$? Give a comment.