

Chapter 1. Probability Theory

Sample space S - All possible outcomes of a particular experiment.

Event A – Subset of S

Probability – $P(A)$. $P(A): S \rightarrow \mathbb{R} \cap [0,1]$

σ - algebra (Definition 1.2.1)

A collection of subsets of S , B , that fulfills

1. $\phi \in B$
2. $A \in B \Rightarrow A^c \in B$
3. $A_1, A_2, \dots \in B \Rightarrow \bigcup_{i=1}^{\infty} A_i \in B$

S finite or countable $\Rightarrow B$ is all subset of S

S not countable for instance. $S = (-\infty, \infty)$. B is all possible intervals of the type (a,b) , $(a, b]$, $[a, b)$, $[a,b]$. (Borel σ - algebra)

Probability function (Definition 1.2.4)

Given S and B , a probability function is a function that satisfies

1. $P(A) \geq 0 \quad \forall A \in B$
2. $P(S) = 1$
3. $\left. \begin{array}{l} A_1, A_2, \dots \in B \\ A_i \cap A_j = \phi, \quad i \neq j \end{array} \right\} \Rightarrow P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

Calculus of probability

1. Addition rule (1.2.9)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

2. Multiplication rule

$$P(A \cap B) = P(A|B) \cdot P(B) \quad (1.3.3)$$

3. The law of total probability (1.2.11)

$S = \bigcup_{i=1}^{\infty} C_i$, $C_i \cap C_j = \phi$, $\forall i \neq j$. Then

$$P(A) = \sum_{i=1}^{\infty} P(A \cap C_i) = \sum_{i=1}^{\infty} P(A|C_i)P(C_i).$$

4. Bayes rule (1.3.5)

$$P(C_i|A) = \frac{P(C_i \cap A)}{P(A)} = \frac{P(A|C_i)P(C_i)}{\sum_{j=1}^{\infty} P(A|C_j)P(C_j)}$$

Independence (1.3.12)

$$P(A \cap B) = P(A) \cdot P(B)$$

Random variables (Definition 1.4.1)

X random variable. $X : S \rightarrow R$

Distribution function (Definition 1.5.1)

$$F_X(x) = P_X(X \leq x), \forall x$$

X is discrete if $F_X(x)$ is a step function

X is continuous if $F_X(x)$ is a continuous function

Probability mass function (X discrete)

$$f_X(x) = P_X(X = x) = P(\{s_j \in s : X(s_j) = x\})$$

$$F_X(a) = \sum_{x \leq a} P_X(X = x)$$

Support of X : All x for which $P_X(X = x) > 0$

Probability density function (X continuous)

$$F_X(x) = \int_{-\infty}^x f_X(t) dt, \forall x$$

$$f_X(x) = \frac{d}{dx} F_X(x) \text{ if } f_X(x) \text{ is continuous.}$$

Support of X : All x for which $f_X(x) > 0$

Identical distributed variables (Definition 1.5.8)

If $P(X \in A) = P(Y \in A) \forall A \in \mathcal{B}$ then X and Y are identical distributed

Chapter 2. Transformations and Expectations

Distributions of Functions of a Random Variable (2.1)

X is defined on \mathcal{X} og $Y = g(X)$ is defined on \mathcal{Y} .

$$P(Y \in A) = P(g(X) \in A) = P(\{x \in \mathcal{X} : g(x) \in A\}) = P(X \in g^{-1}(A))$$

$$g^{-1}(A) = \{x \in \mathcal{X} : g(x) \in A\}$$

$$g^{-1}(y) = \{x \in \mathcal{X} : g(x) = y\}$$

X discrete

$$f_Y(y) = P(Y = y) = \sum_{x \in g^{-1}(y)} P(X = x), \text{ for } y \in \mathcal{Y}.$$

X continuous

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(\{x \in \mathcal{X} : g(x) \leq y\}) = \int_{\{x \in \mathcal{X} : g(x) \leq y\}} f_X(x) dx$$

Monotone transformations (page 50)

g increasing if $u > v \Rightarrow g(u) > g(v)$

g decreasing if $u < v \Rightarrow g(u) < g(v)$

g increasing or decreasing $\Leftrightarrow g$ is monotone.

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|, & y \in Y \\ 0, & \text{elles} \end{cases}$$

Expected Value (2.2)

$$\text{If } \begin{cases} \int_{-\infty}^{\infty} |x| f_X(x) dx < \infty \\ \sum_x |x| P(X=x) < \infty \end{cases} \text{ then } E[X] = \begin{cases} \int_{-\infty}^{\infty} x f_X(x) dx < \infty \\ \sum_x x P(X=x) < \infty \end{cases}$$

Definition 2.2.1

$$E[g(X)] = \begin{cases} \int_{-\infty}^{\infty} g(x) f_X(x) dx \\ \sum_{x \in X} g(x) P(X=x) \end{cases}$$

$$E\left[\sum_{i=1}^n g(X_i)\right] = \sum_{i=1}^n E[g(X_i)]$$

Momentgenerating function (2.3)

$$M_X(t) = E[e^{tX}] = \begin{cases} \int_{-\infty}^{\infty} e^{tx} f_X(x) dx, & X \text{ continuous} \\ \sum_x e^{tx} P(X=x), & X \text{ discrete} \end{cases}$$

$$M_X^n(t) = E[X^n e^{tX}]$$