Problem 1

Let the probability density function (pdf) of a random variable X be given by:

$$f\left(x|\theta\right)=\begin{cases}\theta x^{\theta-1}, & 0<x<1\\ 0, & \text{elsewhere}\end{cases}$$

where $\theta\in\left(0,\infty\right)$.

a) Show that $E\left[X^{k}\right]=\dfrac{\theta}{\theta+k}$, $k>0,$ and use that to find the expected value and the variance of X.

b) Show that $f\left(x|\theta\right)$ is a member of an exponential class of distributions. Show also that the distribution of $Y=-\log X$ is gamma distributed with parameters $\alpha=1$ and $\beta=\dfrac{1}{\theta}$.

c) Find the score statistic for a single random variable X with a pdf given by $f\left(x|\theta\right)$ and show that the Fisher information number is given by $\dfrac{1}{\theta^{2}}$.

d) Let $X_1, X_2, \ldots, X_n$ be independent random variables all with the same distribution as X. Suggest a sufficient statistic for θ that is different from the random sample. Show that the maximum likelihood estimator for θ is given by $\widehat{\theta}=\dfrac{-n}{\sum_{i=1}^{n}\log X_i}$.

e) Find the distribution of $Z=-\sum_{i=1}^{n}\log X_i$. Show that $E\left[Z^{k}\right]=\dfrac{\Gamma\left(n+k\right)}{\theta^{k}\Gamma\left(n\right)}$, $k>-n,$ where $\Gamma\left(.\right)$ is the gamma function and use that to find the expected value and variance of $\widehat{\theta}$.

f) Find the Cramer-Rao lower bound for $Var\left(\hat{\theta}\right)$. What is meant by an efficient estimator for θ?

g) What is the asymptotic distribution of $\hat{\theta}$? Use the asymptotic distribution to show that the interval $\left[\hat{\theta}\left(1-\frac{z_{\frac{\alpha}{2}}}{\sqrt{n}}\right),\ \hat{\theta}\left(1+\frac{z_{\frac{\alpha}{2}}}{\sqrt{n}}\right)\right]$ is an approximate $(1-\alpha)100\%$ confidence interval for θ.

h) Show that $2\theta Z$ (Z is defined in 1e) is chi squared distributed with 2n degrees of freedom and use that to construct an exact $(1-\alpha)100\%$ confidence interval for θ. Compare the length of the intervals when n=10 and $\alpha = 0.05$. You can split α equally putting $\frac{\alpha}{2}$ in each tail of the distribution.

i) Consider now testing $H_0 : \theta = 1$ versus $H_1 : \theta \neq 1$. Show that a likelihood ratio test leads to reject H_0 if $\left(-\sum_{i=1}^{n} \log X_i\right)^n e^{\sum_{i=1}^{n} \log X_i} \leq c$ where c is some constant and use that to show how an exact test for the hypothesis can be constructed based on Z (defined in 1 e).

Problem 2

a) Assume $X_1, X_2, \ldots, X_n$ is a random sample from a $N\left(\theta, \sigma^2\right)$ distribution where $\sigma^2 > 0$ is known. Show that the likelihood function, $L\left(\theta \middle| \boldsymbol{x}\right)$, can be written as:

$$L\left(\theta \middle| \boldsymbol{x}\right) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{\frac{-1}{2\sigma^2}\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2} \cdot e^{\frac{-1}{2\sigma^2}n\left(\bar{x}-\theta\right)^2}$$

Consider now the hypotheses $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ where θ_0 is specified. Let $\lambda\left(\boldsymbol{x}\right)$ be the likelihood ratio for the test. Show that $-2\log\lambda\left(\boldsymbol{X}\right)$ is exact chi squared distributed with one degree of freedom under H_0.