Solution Exercise 10

Problem 1

b) 1. $e^{-\bar{X}}$ is MLE of $\tau(\lambda)$ because of the invariance property of MLEs (see theorem 7.2.10)

It is necessary to compute $\mathrm{E}\left(e^{-\frac{T}{n}}\right)$ and $\mathrm{E}\left(e^{-\frac{2T}{n}}\right)$, where $T = n\bar{\boldsymbol{X}}$. In general, with use of the fact that $T \sim Poisson(n\lambda)$ and the Taylor expansion of the function e^x

$$\mathrm{E}\left(e^{-\frac{aT}{n}}\right) = \sum_{t=0}^{\infty} e^{-\frac{at}{n}} \frac{(n\lambda)^t}{t!} e^{-n\lambda} = e^{-n\lambda} \sum_{t=0}^{\infty} \frac{(e^{-\frac{a}{n}} n\lambda)^t}{t!} = e^{n\lambda(e^{-\frac{a}{n}}-1)}$$

which gives

$$\mathrm{E}\left(e^{-\frac{T}{n}}\right) = e^{n\lambda(e^{-\frac{1}{n}}-1)}$$
$$\mathrm{Var}\left(e^{-\frac{T}{n}}\right) = e^{n\lambda(e^{-\frac{2}{n}}-1)} - e^{2n\lambda(e^{-\frac{1}{n}}-1)}$$

Since $e^{-\frac{1}{n}} - 1 \approx -\frac{1}{n} + \frac{1}{2n^2}$ and $e^{-\frac{2}{n}} - 1 \approx -\frac{2}{n} + \frac{2}{n^2}$,

$$\mathrm{E}\left(e^{-\frac{T}{n}}\right) \approx e^{-\lambda} e^{\frac{\lambda}{2n}}$$
$$\mathrm{Var}\left(e^{-\frac{T}{n}}\right) \approx e^{-2\lambda+\frac{\lambda}{n}}(e^{\frac{\lambda}{n}} - 1) \approx \frac{\lambda}{n} e^{-2\lambda+\frac{\lambda}{n}}$$

2. Analogously to the previous case,

$$\mathrm{E}\left(\left(1-\frac{1}{n}\right)^T\right) = \sum_{t=1}^{\infty}\left(1-\frac{1}{n}\right)^t \frac{(n\lambda)^t}{t!} e^{-n\lambda} = e^{-n\lambda} \sum_{t=1}^{\infty} \frac{((n-1)\lambda)^t}{t!} = e^{-n\lambda} e^{(n-1)\lambda} = e^{-\lambda}$$
$$\mathrm{E}\left(\left(1-\frac{1}{n}\right)^{2T}\right) = \sum_{t=1}^{\infty}\left(1-\frac{1}{n}\right)^{2t} \frac{(n\lambda)^t}{t!} e^{-n\lambda} = e^{-2\lambda+\frac{\lambda}{n}}$$

which gives

$$\mathrm{Var}\left(\left(1-\frac{1}{n}\right)^T\right) = e^{-2\lambda}(e^{\frac{\lambda}{n}} - 1) \approx \frac{\lambda}{n} e^{-2\lambda}$$

Note, that the estimator e^{-nT} is the limit of the estimator $\left(1-\frac{1}{n}\right)^T$.

Problem 2

7.38 Use Corollary 7.3.15.

a.

$$\begin{aligned}\frac{\partial}{\partial\theta}\log L(\theta|\mathbf{x}) &= \frac{\partial}{\partial\theta}\log\prod_i \theta x_i^{\theta-1} = \frac{\partial}{\partial\theta}\sum_i[\log\theta+(\theta-1)\log x_i] \\ &= \sum_i\left[\frac{1}{\theta}+\log x_i\right] = -n\left[-\sum_i\frac{\log x_i}{n}-\frac{1}{\theta}\right].\end{aligned}$$

Thus, $-\sum_i \log X_i/n$ is the UMVUE of $1/\theta$ and attains the Cramér-Rao bound.

b.

$$\begin{aligned}\frac{\partial}{\partial\theta}\log L(\theta|\mathbf{x}) &= \frac{\partial}{\partial\theta}\log\prod_i\frac{\log\theta}{\theta-1}\theta^{x_i} = \frac{\partial}{\partial\theta}\sum_i[\log\log\theta-\log(\theta-1)+x_i\log\theta] \\ &= \sum_i\left(\frac{1}{\theta\log\theta}-\frac{1}{\theta-1}\right)+\frac{1}{\theta}\sum_i x_i = \frac{n}{\theta\log\theta}-\frac{n}{\theta-1}+\frac{n\bar{x}}{\theta} \\ &= \frac{n}{\theta}\left[\bar{x}-\left(\frac{\theta}{\theta-1}-\frac{1}{\log\theta}\right)\right].\end{aligned}$$

Thus, $\bar{X}$ is the UMVUE of $\frac{\theta}{\theta-1}-\frac{1}{\log\theta}$ and attains the Cramér-Rao lower bound.

Note: We claim that if $\frac{\partial}{\partial\theta}\log L(\theta|\mathbf{X}) = a(\theta)[W(\mathbf{X})-\tau(\theta)]$, then $\mathrm{E}\,W(\mathbf{X}) = \tau(\theta)$, because under the condition of the Cramér-Rao Theorem, $\mathrm{E}\,\frac{\partial}{\partial\theta}\log L(\theta|\mathbf{x}) = 0$. To be rigorous, we need to check the “interchange differentiation and integration“ condition. Both (a) and (b) are exponential families, and this condition is satisfied for all exponential families.

Problem 3

7.47 $X_i \sim \mathrm{n}(r, \sigma^2)$, so $\bar{X} \sim \mathrm{n}(r, \sigma^2/n)$ and $\mathrm{E}\,\bar{X}^2 = r^2 + \sigma^2/n$. Thus $\mathrm{E}\left[(\pi\bar{X}^2 - \pi\sigma^2/n)\right] = \pi r^2$ is best unbiased because $\bar{X}$ is a complete sufficient statistic. If σ^2 is unknown replace it with s^2 and the conclusion still holds.

Problem 4

6.18 The distribution of $Y = \sum_i X_i$ is Poisson$(n\lambda)$. Now

$$\mathrm{E}g(Y) = \sum_{y=0}^{\infty} g(y) \frac{(n\lambda)^y e^{-n\lambda}}{y!}.$$

If the expectation exists, this is an analytic function which cannot be identically zero.

for all values of λ .

Problem 5

7.52 a. Because the Poisson family is an exponential family with $t(x) = x$, $\sum_i X_i$ is a complete sufficient statistic. Any function of $\sum_i X_i$ that is an unbiased estimator of λ is the unique best unbiased estimator of λ. Because $\bar{X}$ is a function of $\sum_i X_i$ and $\mathrm{E}\,\bar{X} = \lambda$, $\bar{X}$ is the best unbiased estimator of λ.

b. S^2 is an unbiased estimator of the population variance, that is, $\mathrm{E}\,S^2 = \lambda$. $\bar{X}$ is a one-to-one function of $\sum_i X_i$. So $\bar{X}$ is also a complete sufficient statistic. Thus, $\mathrm{E}(S^2|\bar{X})$ is an unbiased estimator of λ and, by Theorem 7.3.23, it is also the unique best unbiased estimator of λ. Therefore $\mathrm{E}(S^2|\bar{X}) = \bar{X}$. Then we have

$$\mathrm{Var}\,S^2 = \mathrm{Var}\left(\mathrm{E}(S^2|\bar{X})\right) + \mathrm{E}\,\mathrm{Var}(S^2|\bar{X}) = \mathrm{Var}\,\bar{X} + \mathrm{E}\,\mathrm{Var}(S^2|\bar{X}),$$

so $\mathrm{Var}\,S^2 > \mathrm{Var}\,\bar{X}$.

c. We formulate a general theorem. Let $T(X)$ be a complete sufficient statistic, and let $T'(X)$ be any statistic other than $T(X)$ such that $\mathrm{E}\,T(X) = \mathrm{E}\,T'(X)$. Then $\mathrm{E}[T'(X)|T(X)] = T(X)$ and $\mathrm{Var}\,T'(X) > \mathrm{Var}\,T(X)$.