TMA4295 Statistical inference
Exercise 11 - solution

Problem 1

$X_1, ..., X_n$ i.i.d. $N(\mu, \sigma^2)$, we want to test

$$H_0 : \sigma^2 = \sigma_0^2 \ vs \ H_1 : \sigma^2 \neq \sigma_0^2$$

a) $\lambda(x, y) = \frac{\sup_{\Theta_0} L(\theta|x)}{\sup_{\Theta} L(\theta|x)}$, so we need to find the the values of θ that maximize L under H_0 and for the general case.

Here the parameter is $\theta = (\mu, \theta)$, for the general case the values that maximize L are obtained with the maximum likelihood estimation

$$\hat{\mu} = \bar{X}$$

$$\hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(X - \hat{\mu})^2.$$

While under H_0, the parameter space is $\Theta_0 = \{(\mu, \sigma_0^2) : \infty < \mu < \infty\}$. The likelihood function for the Gaussian distribution is given by $L(\mu, \sigma^2) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left(-\sum_{i=1}^n \frac{(x_i-\mu)^2}{2\sigma^2}\right)$, and under H_0 (so with $\sigma = \sigma_0$) it is easy to verify that the maximum in L is reached with

$$\hat{\mu}_0 = \bar{X}.$$

Then

$$\lambda(x, y) = \frac{\left(\frac{1}{\sqrt{2\pi}\sigma_0}\right)^n \exp\left(-\sum_{i=1}^n \frac{(x_i-\bar{X})^2}{2\sigma_0^2}\right)}{\left(\frac{1}{\sqrt{2\pi\frac{1}{n}\sum_{i=1}^n(X-\hat{\mu})^2}}\right)^n \exp\left(-\sum_{i=1}^n \frac{(x_i-\bar{X})^2}{2\frac{1}{n}\sum_{i=1}^n(X-\hat{\mu})^2}\right)} = \left(\frac{Z}{n}\right)^{n/2} \exp((n - Z)/2)$$

with $Z = \frac{n\hat{\sigma}^2}{\sigma_0^2}$.

b) Note that

$$-2 \ln \lambda(\mathbf{X}) = -2\left(\frac{n}{2} \ln\left(\frac{Z}{n}\right) + \frac{n - Z}{2}\right) = Z - n - n \ln \frac{Z}{n} = g(Z).$$

Looking at the plot of $g(z)$ we observe that the rejection region is then given by $R = \{z : g(z) \geq -2 \ln c\} = \{z : z - n - n \ln \frac{z}{n} \geq -2 \ln c\} = \{Z \leq z_0 \ or \ Z \geq z_1\}$.

c) We can notice that with $z_0 = 3.18$ and $z_1 = 22.91$ we have $g(z_0) = g(z_1)$ and from the cumulative distribution function of a χ_9^2 we have that $P(Z \leq z_0) + P(Z \geq z_1) = 0.05$

d) Here the main difference is that I'm looking at $g(Z)$ instead of Z. Hence the rejection region is $R = \{z : g(z) > -2 \ln c\} = \{z : g(z) > C\}$. Since $g(Z) \sim \chi_1^2$ and $\alpha = 0.05 = P(Z \in R) = P(g(Z) \geq C)$ we get $C = 3.84$. Then it will be enough to find z_0 and z_1 such that $g(z_0) = g(z_1) = 3.84$

e) Using the Central Limit Theorem we have

$$\frac{Z - (n - 1)}{\sqrt{2(n - 1)}} \to N(0, 1) \quad \text{in distribution}$$

now we notice that

$$U_n = \frac{Z - n}{\sqrt{(2n)}} = \frac{\sqrt{2(n-1)}}{\sqrt{2n}} \frac{Z - (n - 1)}{\sqrt{2(n - 1)}} - \frac{1}{\sqrt{2n}} \to N(0, 1) \quad \text{in distribution.}$$

Since

$$-2\ln\lambda(\mathbf{X}) = \sqrt{2n}U_n - n\ln\left(\frac{U_n\sqrt{2}}{\sqrt{n}}+1\right) \approx \sqrt{2n}U_n - n\left(\frac{U_n\sqrt{2}}{\sqrt{n}} - \frac{1}{2}\frac{2U_n^2}{n}\right) = U_n^2$$

and $U_n \sim \chi_1^2$ we have that $-2\ln\lambda(\mathbf{X}) \sim \chi_1^2$.

Problem 2

$X \sim Poisson(\alpha)$ and $Y \sim Poisson(\beta)$

a) $L(\alpha,\beta|x,y) = \frac{\alpha^x}{x!}e^{-\alpha}\frac{\beta^y}{y!}e^{-\beta}$. The MLE for α and β in the full model can be computed by taking the first derivatives respect to α and β of the log-likelihood and set the derivatives equal to zero.

$$\hat{\alpha} = x,$$
$$\hat{\beta} = y.$$

The MLE for α and β under H_0 can be computed as we did previously, keeping in mind that under H_0 we have $\alpha = \beta$.

$$\hat{\alpha}_0 = \frac{x+y}{2},$$
$$\hat{\beta}_0 = \frac{x+y}{2}.$$

b)

$$\lambda(x,y) = \frac{L(\hat{\alpha}_0,\hat{\beta}_0|x,y)}{L(\hat{\alpha},\hat{\beta}|x,y)} = \frac{\frac{\left(\frac{x+y}{2}\right)^x}{x!}e^{-\left(\frac{x+y}{2}\right)}\frac{\left(\frac{x+y}{2}\right)^y}{y!}e^{-\left(\frac{x+y}{2}\right)}}{\frac{x^x}{x!}e^{-x}\frac{y^y}{y!}e^{-y}} = \left(\frac{x+y}{2x}\right)^x\left(\frac{x+y}{2x}\right)^y.$$

Problem 3

a) $X_1,...,X_n$ i.i.d $f(x-\mu)$ and we want to prove that $\bar{X}-\mu$ is a pivot.

Consider $X_i = (X_i-\mu)+\mu = Z_i+\mu$, then we have $\bar{X}-\mu = \bar{Z}+\mu-\mu = \bar{Z}$. It is easy to verify that the distribution of Z_i does not depend on μ, and so also $\bar{Z}$. Hence $\bar{X}-\mu$ is a pivot.

b) $X_1,...,X_n$ i.i.d $\frac{1}{\sigma}f\left(\frac{x}{\sigma}\right)$ and we want to prove that $\frac{\bar{X}}{\sigma}$ is a pivot.

Consider $X_i = \sigma\frac{X_i}{\sigma} = \sigma Z_i$, then we have $\frac{\bar{X}}{\sigma} = \sigma\frac{\bar{Z}}{\sigma} = \bar{Z}$. It is easy to verify that the distribution of Z_i does not depend on σ, and so also $\bar{Z}$. Hence $\frac{\bar{X}}{\sigma}$ is a pivot.

c) $X_1,...,X_n$ i.i.d $\frac{1}{\sigma}f\left(\frac{x-\mu}{\sigma}\right)$ and we want to prove that $\frac{\bar{X}-\mu}{S}$ is a pivot.

consider $X_i = (X_i-\mu)+\mu = \sigma\frac{(X_i-\mu)}{\sigma}+\mu = \sigma Z_i+\mu$ then $\bar{X}-\mu = \sigma\bar{Z}$ and $S^2 = \frac{1}{1-n}\sum_{i=1}^n(\sigma Z_i - \sigma\bar{Z})^2 = \sigma^2 S_Z^2$. This implies that $\frac{\bar{X}-\mu}{S} = \frac{\sigma\bar{Z}}{\sigma S_Z}$. It is easy to verify that the distribution of Z_i does not depend on μ and σ, hence also $\bar{Z}$. Then $\frac{\bar{X}-\mu}{S}$ is a pivot.