TMA4295 Statistical inference
Exercise 12 - solution

Problem 1

Using theorem 2.1.10 we have that $F_T(T|\theta)$ is uniform(0,1) and is a pivot. Hence we have

$$P_{\theta_0}\left(\{T : \alpha_1 \leq F_T(T|\theta_0) \leq 1-\alpha_2\}\right) = P(\alpha_1 \leq U \leq 1-\alpha_2) = 1-\alpha_1-\alpha_2 = 1-\alpha$$

where $U \sim uniform(0,1)$. Then we have as an α level acceptance region

$$\{t : \alpha_1 \leq \ F_T(t|\theta_0) \leq 1-\alpha_2\}$$

and as $\alpha - 1$ confidence interval

$$\{\theta : \alpha_1 \leq \ F_T(t|\theta) \leq 1-\alpha_2\}.$$

Problem 2

The confidence interval derived by the method of Section 9.2.3 is

$$C(y) = \left\{ \mu \colon y + \frac{1}{n} \log\left(\frac{\alpha}{2}\right) \le \mu \le y + \frac{1}{n}\log\left(1 - \frac{\alpha}{2}\right) \right\}$$

where $y = \min_i x_i$. The LRT method derives its interval from the test of $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$. Since Y is sufficient for μ, we can use $f_Y(y \mid \mu)$. We have

$$\begin{aligned}
\lambda(y) &= \frac{\sup_{\mu=\mu_0} L(\mu|y)}{\sup_{\mu\in(-\infty,\infty)} L(\mu|y)} &= \frac{ne^{-n}(y-\mu_0)I_{[\mu_0,\infty)(y)}}{ne^{-(y-y)}I_{[\mu,\infty)(y)}} \\
&= e^{-n(y-\mu_0)}I_{[\mu_0,\infty)}(y) &= \begin{cases} 0 & \text{if } y < \mu_0 \\ e^{-n(y-\mu_0)} & \text{if } y \ge \mu_0. \end{cases}
\end{aligned}$$

We reject H_0 if $\lambda(y) = e^{-n(y-\mu_0)} < c_\alpha$, where $0 \le c_\alpha \le 1$ is chosen to give the test level α. To determine c_α, set

$$\begin{aligned}
\alpha &= P\{\text{reject } H_0 | \, \mu = \mu_0\} &= P\left\{ Y > \mu_0 - \frac{\log c_\alpha}{n} \text{ or } Y < \mu_0 \middle| \, \mu = \mu_0 \right\} \\
&= P\left\{ Y > \mu_0 - \frac{\log c_\alpha}{n} \middle| \, \mu = \mu_0 \right\} &= \int_{\mu_0 - \frac{\log c_\alpha}{n}}^{\infty} ne^{-n(y-\mu_0)}\, dy \\
&= -e^{-n(y-\mu_0)}\Big|_{\mu_0-\frac{\log c_\alpha}{n}}^{\infty} = e^{\log c_\alpha} &= c_\alpha.
\end{aligned}$$

Therefore, $c_\alpha = \alpha$ and the $1-\alpha$ confidence interval is

$$C(y) = \left\{ \mu \colon \mu \le y \le \mu - \frac{\log \alpha}{n} \right\} = \left\{ \mu \colon y + \frac{1}{n}\, \log \alpha \le \mu \le y \right\}.$$

To use the pivotal method, note that since μ is a location parameter, a natural pivotal quantity is $Z = Y - \mu$. Then, $f_Z(z) = ne^{-nz}I_{(0,\infty)}(z)$. Let $P\{a \le Z \le b\} = 1 - \alpha$, where a and b satisfy

$$\begin{aligned}
\frac{\alpha}{2} = \int_0^a ne^{-nz}\, dz = -e^{-nz}\big|_0^a = 1 - e^{-na} \quad &\Rightarrow \quad e^{-na} = 1 - \frac{\alpha}{2} \\
&\Rightarrow \quad a = \frac{-\log\left(1 - \frac{\alpha}{2}\right)}{n} \\
\frac{\alpha}{2} = \int_b^\infty ne^{-nz}\, dz = -e^{-nz}\big|_b^\infty = e^{-nb} \quad &\Rightarrow \quad -nb = \log\frac{\alpha}{2} \\
&\Rightarrow \quad b = -\frac{1}{n}\log\left(\frac{\alpha}{2}\right)
\end{aligned}$$

Thus, the pivotal interval is $Y + \log(\alpha/2)/n \le \mu \le Y + \log(1-\alpha/2)$, the same interval as from Example 9.2.13. To compare the intervals we compare their lengths. We have

$$\begin{aligned}
\text{Length of LRT interval} &= y - (y + \frac{1}{n}\log\alpha) = -\frac{1}{n}\log\alpha \\
\text{Length of Pivotal interval} &= \left(y + \frac{1}{n}\log(1-\alpha/2)\right) - (y + \frac{1}{n}\log\alpha/2) = \frac{1}{n}\log\frac{1-\alpha/2}{\alpha/2}
\end{aligned}$$

Thus, the LRT interval is shorter if $-\log\alpha < \log[(1-\alpha/2)/(\alpha/2)]$, but this is always satisfied.

Problem 3

Solution. The likelihood function is

$$L(\theta; X_1, ..., X_n) = \theta^{n\theta}[\Gamma(\theta)]^{-n}(X_1 \cdot ... \cdot X_n)^{\theta-1}e^{-\theta \sum X_i} =$$

$$= \theta^{n\theta}[\Gamma(\theta)]^{-n}(X_1 \cdot ... \cdot X_n e^{-\sum X_i})^{\theta}(X_1 \cdot ... \cdot X_n)^{-1}.$$

Put

$$T(X_1, ..., X_n) = \prod X_i \cdot e^{-\sum X_i},$$

$$g(T, \theta) = \theta^{n\theta}[\Gamma(\theta)]^{-n}[T(X_1, ..., X_n)]^{\theta},$$

and

$$h(X_1, ..., X_n) = (X_1 \cdot ... \cdot X_n)^{-1}.$$

Problem 4

a) Denote the Fisher information of the sample and that of one observation by $I(\theta)$ and $I_0(\theta)$ respectively. Then $I(\theta) = nI_0(\theta)$ and

$$I_0(\theta) = E\left(\frac{\partial \ln f(X;\theta)}{\partial \theta}\right)^2 = E\left(\frac{X^2}{\theta^3} - \frac{1}{\theta}\right)^2 = \frac{1}{\theta^6}EX^4 - \frac{1}{\theta^2}$$

To find EX^4 we can use mgf: $EX^n = {M_X}^{(n)}(0)$.

We have

$$M_X(t) = e^{\theta^2 t^2/2}$$

$$M_X''(t) = e^{\theta^2 t^2/2}(\theta^4 t^2 + \theta^2)$$

$$M_X'''(t) = e^{\theta^2 t^2/2}(\theta^6 t^3 + 3\theta^4 t)$$

$$M_X^{(4)}(t) = e^{\theta^2 t^2/2}(\theta^2 t)(\theta^6 t^3 + 3\theta^4 t) + e^{\theta^2 t^2/2}(3\theta^6 t^2 + 3\theta^4)$$

Hence $EX^4 = M^{(4)}(0) = 3\theta^4$

and

$$I_0(\theta) = \frac{2}{\theta^2}, \quad I(\theta) = \frac{2n}{\theta^2}$$

b)

$$ET_n = \frac{2}{n}EX_1^2 + \frac{(n-2)}{n(n-1)}\sum_{i=2}^{n} EX_i^2 = \theta^2\left(\frac{2}{n} + \frac{(n-2)}{n(n-1)}(n-1)\right) = \theta^2$$

i.e. T_n is unbiased.

c) It is consistent: $\frac{2}{n}X_1^2 \xrightarrow{P} 0,$

$$\frac{n-2}{n} \longrightarrow 1, \qquad \frac{1}{n-1}\sum_{i=2}^{n} X_i^2 \xrightarrow{P} EX^2 = \theta^2$$

d)

$$VarT_n = \frac{4}{n^2}Var(X_1^2) + \frac{(n-2)^2}{n^2(n-1)^2}\sum_{i=2}^{n} Var(X_i^2) =$$

$$= Var(X^2)\left[\frac{4}{n^2} + \frac{(n-2)^2(n-1)}{n^2(n-1)^2}\right] = Var(X^2)\frac{1}{n-1} = \frac{2\theta^4}{n-1}$$

since

$$Var(X^2) = EX^4 - (EX^2)^2 = 3\theta^4 - \theta^4 = 2\theta^4$$

(EX^4 was obtained in part (a)).

The Cramer-Rao lower bound is (use part (a))

$$\frac{\left[\frac{d}{d\theta}(\theta^2)\right]^2}{I(\theta)} = \frac{4\theta^2}{2n/\theta^2} = \frac{2\theta^4}{n} < \frac{2\theta^4}{n-1} = Var(T_n).$$

Hence T_n is not efficient

e)

$$L(\theta; X_1, ..., X_n) = (2\pi)^{-n/2}\theta^{-n}e^{-\frac{1}{2\theta^2}\sum X_i^2}.$$

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$$\frac{\partial \ln L}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{\theta^3}\sum X_i^2.$$

Problem 5

a) The acceptance region: $\lambda(X) \geq c$ where

$$\lambda(X) = \frac{L(\theta_0; X)}{\sup L(\theta; X)} = \frac{L(\theta_0; X)}{L(\hat{\theta}_{MLE}; X)},$$

and c is found from the condition

$$P_{\theta_0}(\lambda(X) \geq c) = 1 - \alpha$$

$$\hat{\theta}_{MLE} = \frac{1}{n}\sum X_i = \bar{X}$$

$$\lambda(X) = e^{-\frac{n}{2}(\bar{X}-\theta_0)^2}$$

So, the acceptance region

$$\theta_0 - \frac{1}{\sqrt{n}}l_{1-\frac{\alpha}{2}} \leq \bar{X} \leq \theta_0 + \frac{1}{\sqrt{n}}l_{1-\frac{\alpha}{2}}$$

where l_δ - δ-quantile of the standard normal distribution.

b) Inverting the test of part (a) we obtain the following $(1-\alpha)$ confidence interval:

$$\left[\bar{X} - \frac{1}{\sqrt{n}}l_{1-\frac{\alpha}{2}}, \bar{X} + \frac{1}{\sqrt{n}}l_{1-\frac{\alpha}{2}}\right].$$