Solution TMA 4295 H-17

1a) $$E[X^k] = \int_0^1 \theta x^{\theta+k-1}\,dx = \theta\left[\frac{x^{\theta+k}}{\theta+k}\right]_0^1 = \frac{\theta}{\theta+k}$$

Hence $$E[X] = \frac{\theta}{\theta+1}$$

$$Var[X] = \frac{\theta}{\theta+2} - \frac{\theta^2}{(\theta+1)^2} = \frac{\theta(\theta+1)^2 - \theta^2(\theta+2)}{(\theta+1)^2(\theta+2)} = \underline{\frac{\theta}{(\theta+1)^2(\theta+2)}}$$

b) $$f(x|\theta) = \theta x^{\theta-1} I_{[0,1]} = \theta e^{(\theta-1)\log x} I_{[0,1]} = \frac{1}{x}\cdot\theta e^{\theta\log x} I_{[0,1]}$$

$$= h(x)\, c(\theta)\, e^{w(\theta)t(x)} \quad \text{where } h(x) = \frac{1}{x} I_{[0,1]} > 0,\ c(\theta) = \theta > 0,\ w(\theta) = \theta,\ t(x) = \log x$$

$$Y = -\log X \Rightarrow X = e^{-Y}$$

$$\Rightarrow f_Y(y|\theta) = \theta e^{-y(\theta-1)} \cdot e^{-y} = \theta e^{-\theta y}, \quad y \in (0,\infty)$$

$$f_Y(y|\theta) = \frac{1}{\Gamma(1)}\cdot\frac{1}{\frac{1}{\theta}} y^{1-1} e^{-\frac{y}{\frac{1}{\theta}}} \Rightarrow \text{gamma}(1, \tfrac{1}{\theta})$$

c) $$\log f(X|\theta) = \log\theta + (\theta-1)\log X$$

$$\frac{d}{d\theta}\log f(X|\theta) = \frac{1}{\theta} + \log X \Rightarrow S(X|\theta) = \frac{1}{\theta} + \log X$$

$$I_X(\theta) = Var[\log X] = Var[Y] = 1\cdot\left(\frac{1}{\theta}\right)^2 = \frac{1}{\theta^2}$$

or $$-E\left[\frac{d}{d\theta} S(X|\theta)\right] = \frac{1}{\theta^2}$$

d) $$f(x_1,\dots,x_n|\theta) = \prod_{i=1}^n f(x_i|\theta) = \theta^n e^{(\theta-1)\sum_{i=1}^n \log x_i} = \underbrace{e^{-\sum_{i=1}^n \log x_i}}_{h(\underline{x})} \cdot \underbrace{\theta^n e^{\theta\sum_{i=1}^n \log x_i}}_{g(T(\underline{x})|\theta)}$$

$$\Rightarrow \sum_{i=1}^n \log X_i \quad \left(\text{or } \prod_{i=1}^n X_i\right) \text{ is a sufficient statistic}$$

$$L(\theta|x_1,\dots,x_m) = \theta^m e^{(\theta-1)\sum_{i=1}^m \log x_i} \Rightarrow \log L(\theta|x_1,\dots,x_m) = m\log\theta + (\theta-1)\sum_{i=1}^m \log x_i$$

$$\frac{d}{d\theta}\log L(\theta|x_1,\dots,x_m) = \frac{m}{\theta} + \sum_{i=1}^m \log x_i = 0 \Rightarrow \theta = \frac{-m}{\sum_{i=1}^n \log x_i}$$

$$\frac{d^2}{d\theta^2}\log L(\theta|x_1,\dots,x_m) = -\frac{m}{\theta^2} < 0 \quad \forall\theta \Rightarrow \hat{\theta} = \frac{-m}{\sum_{i=1}^n \log x_i} = \text{MLE}$$

e)

$$-\log X_i \sim \text{gamma}\left(1, \frac{1}{\theta}\right), \quad i = 1, 2, \dots, m$$

$$\Rightarrow Z = -\sum_{i=1}^m \log X_i \sim \text{gamma}\left(\sum_{i=1}^m 1, \frac{1}{\theta}\right) = \text{gamma}\left(m, \frac{1}{\theta}\right)$$

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$$f(z|\theta) = \frac{1}{\Gamma(m)} \frac{z^{m-1}}{\left(\frac{1}{\theta}\right)^m} e^{-\theta z} \Rightarrow E[Z^k] = \int_0^\infty \frac{1}{\Gamma(m)} z^{m+k-1} \theta^m e^{-\theta z} dz$$

$$\underset{=}{t=\theta z} \int_0^\infty \frac{1}{\Gamma(m)} \theta^m \frac{t^{m+k-1}}{\theta^{m+k}} e^{-t} dt = \frac{1}{\theta^k \Gamma(m)} \int_0^\infty t^{m+k-1} e^{-t} dt = \frac{\Gamma(m+k)}{\theta^k \Gamma(m)}$$

$$E[\hat{\theta}] = m E[Z^{-1}] = \frac{m\Gamma(m-1)}{\frac{1}{\theta}\Gamma(m)} = \frac{\theta m}{m-1}$$

$$Var[\hat{\theta}] = \theta^2\left[\frac{m^2\Gamma(m-2)}{\Gamma(m)} - \frac{m^2}{(m-1)^2}\right] = \theta^2\left[\frac{m^2}{(m-1)(m-2)} - \frac{m^2}{(m-1)^2}\right] = \frac{\theta^2 m^2}{(m-1)^2(m-2)}$$

f)

$$\frac{d}{d\theta}\left[E[\hat{\theta}]\right] = \frac{m}{m-1} \Rightarrow \text{Cramer-Rao lower bound}$$

$$= \frac{\left(\frac{m}{m-1}\right)^2}{m \cdot \frac{1}{\theta^2}} = \frac{\theta^2 m}{(m-1)^2} < \frac{\theta^2 m^2}{(m-1)^2(m-2)}$$

Efficient finite sample:

$\hat{\theta}$ is unbiased and attains its lower bound.

Asymptotically efficient

$\sqrt{n}[\hat{\theta} - \theta] \xrightarrow{D} n(0, v(\theta))$ where $v(\theta)$ is the Fisher information number for one observation.

$\hat{\theta}$ is asymptotically efficient.

g)

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{D} n(0, v(\theta)) \quad \text{where } v(\theta) = \frac{1}{I_X(\theta)} = \frac{1}{\frac{1}{\theta^2}} = \theta^2$$

$$\Rightarrow \frac{\hat{\theta} - \theta}{\frac{\theta}{\sqrt{n}}} \xrightarrow{D} N(0,1).$$

Approximating θ with $\hat{\theta}$ (the variance of $\hat{\theta} \to 0$ as $n \to \infty$) gives

$$P\left(-z_{\frac{\alpha}{2}} \le \frac{\hat{\theta} - \theta}{\frac{\hat{\theta}}{\sqrt{n}}} \le z_{\frac{\alpha}{2}}\right) \approx 1 - \alpha \text{ for large } n$$

and the $(1-\alpha)100\%$ confidence interval: $\left[\hat{\theta}\left(1 - \frac{z_{\frac{\alpha}{2}}}{\sqrt{n}}\right), \hat{\theta}\left(1 + \frac{z_{\frac{\alpha}{2}}}{\sqrt{n}}\right)\right]$

h)

$$U = 2\theta Z \Rightarrow f(u) = \frac{1}{\Gamma(n)} \frac{u^{n-1}}{(2\theta)^{n-1}} \theta^n e^{-\frac{\theta u}{2\theta}} \cdot \frac{1}{2\theta}$$

$$= \frac{1}{\Gamma(n)} \frac{u^{n-1}}{2^n} e^{-\frac{u}{2}}, \quad u > 0 \quad \text{which is gamma}(n, 2) \text{ or } \chi^2(2n)$$

Thereby $P\left(\chi^2(2n)_{,1-\frac{\alpha}{2}} \le 2\theta Z \le \chi^2(2n)_{,\frac{\alpha}{2}}\right) = 1 - \alpha$

$$\Leftrightarrow P\left(\frac{\chi^2(2n)_{,1-\frac{\alpha}{2}}}{2Z} \le \theta \le \frac{\chi^2(2n)_{,\frac{\alpha}{2}}}{2Z}\right) = 1 - \alpha$$

Length of interval in g: $\frac{2\hat{\theta}(z_{\frac{\alpha}{2}})}{\sqrt{n}} = \frac{2n}{Z}\left(\frac{z_{\frac{\alpha}{2}}}{\sqrt{n}}\right) = \frac{2\sqrt{n}}{Z} \cdot z_{\frac{\alpha}{2}} = \frac{12.4}{Z}$

Length of interval in h: $\frac{\chi^2(2n)_{\frac{\alpha}{2}} - \chi^2(2n)_{,1-\frac{\alpha}{2}}}{2Z} = \frac{34.17 - 9.59}{2Z} = \frac{12.29}{Z}$

The exact interval in h is shorter (just a little)

i)

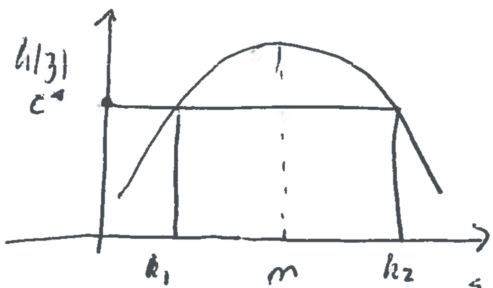
$$\lambda(\underline{x}) = \frac{\sup_{\theta=1} \theta^n e^{(\theta-1)\sum_1^n \log x_i}}{\sup_{\theta} \theta^n e^{(\theta-1)\sum_1^n \log x_i}} = \frac{1}{\hat{\theta}^n e^{-n} e^{-\sum_1^n \log x_i}}$$

$$= \frac{\left(-\sum_1^n \log x_i\right)^n \cdot e^{\sum \log x_i}}{n^n e^{-n}} \le c \iff \left(-\sum_1^n \log x_i\right)^n e^{\sum_1^n \log x_i} \le c n^n e^{-n} = c^*$$

Introduce $Z = -\sum_1^n \log x_i$ and we shall reject H_0 if.

~~$h(z) = z^n e^{-z} \le c$.~~ $h(z) = z^n e^{-z} \le c^*$

$\frac{d}{dz} h(z) = z^{n-1} e^{-z}(n-z)$ which shows that $h(z)$ has a single maximum for $z = n$. Hence we reject H_0 if.



$Z \le k_1$ and $Z \ge k_2$

Under H_0 $2Z \sim \chi^2(2n)$ so k_1 and k_2 must satisfy:

① $k_1^n e^{-k_1} = k_2^n e^{-k_2}$ or $\left(\frac{k_1}{k_2}\right)^n = \frac{e^{-k_1}}{e^{-k_2}}$

② $P(2k_1 \le 2Z \le 2k_2) = 1-\alpha$.

Problem 2a

$$L(\theta|\underline{x}) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_1^n (x_i-\theta)^2}$$

$$\sum_1^n (x_i-\theta)^2 = \sum_1^n (x_i - \bar{x} + \bar{x} - \theta)^2 = \sum_1^n (x_i - \bar{x})^2 + \sum_1^n (\bar{x}-\theta)^2 \text{ since } \sum_1^n (x_i - \bar{x}) = 0$$

$$\sum_1^n (\bar{x}-\theta)^2 = n(\bar{x}-\theta)^2 \Rightarrow L(\theta|x) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_1^n (x_i-\bar{x})^2} \cdot e^{-\frac{1}{2\sigma^2} n(\bar{x}-\theta)^2}$$

$$\lambda(\underline{x}) = \frac{\sup_{\theta=\theta_0} L(\theta|x)}{\sup_{\theta} L(\theta|x)} = \frac{\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_1^n (x_i-\bar{x})^2} \cdot e^{-\frac{1}{2\sigma^2} n(\bar{x}-\theta_0)^2}}{\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_1^n (x_i-\bar{x})^2} \cdot e^{-\frac{1}{2\sigma^2} n(\bar{x}-\bar{x})}} \quad \text{since } \hat{\theta} = \bar{x}$$

$$= e^{-\frac{1}{2\sigma^2} n(\bar{x}-\theta_0)^2} \Rightarrow -2\log\lambda(x) = \frac{(\bar{x}-\theta_0)^2}{\sigma^2/n} \sim \chi^2(1) \text{ since } \frac{\bar{x}-\theta_0}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$