**Confidence interval in the Poisson distribution.**

$X_1,\ldots,X_n$ iid Poisson$\left(\lambda\right)$.

$Y=\sum_{i=1}^{n}X_i \sim Poisson\left(n\lambda\right)$ and sufficient.

Observe $Y=y_0$

Pivoting cdf: $P\left(Y\le y_0\middle|\lambda_U\left(y_0\right)\right)=\frac{\alpha}{2}=P\left(Y\ge y_0\middle|\lambda_L\left(y_0\right)\right)$

Let $Z\sim gamma\left(\alpha^*,\beta\right)$

$P\left(Z\le z\right)=P\left(Y\ge\alpha^*\right)$

$Z=\sum_{i=1}^{\alpha^*}E_i$, where $E_i=\exp\left(\frac{1}{\beta}\right)=\exp\left(\lambda\right)$

For $\lambda_U\left(y_0\right)$ we get

$\frac{\alpha}{2}=P\left(Y\le y_0\right)=1-P\left(Y\ge y_0+1\right)=1-P\left(Z\le z\right)=P\left(Z>z\right)$

Where $Z\sim$ gamma$\left(y_0+1,\ \frac{1}{\lambda}\right)\Rightarrow 2\lambda Z\sim\chi^2\left(2\left(y_o+1\right)\right)$

Z represents the time, and $\frac{1}{\lambda}$ is the expected time between events. Thereby $\frac{z}{\frac{1}{\lambda}} = n\lambda$ or $z = n$.

Hence

$$\frac{\alpha}{2} = P(Z > z) = P(2\lambda Z > 2\lambda z) = P\left(\chi^2\left(2(y_0+1)\right) > 2\lambda n\right) \Rightarrow \lambda_U(t) = \frac{1}{2n}\chi^2\left(2(y_0+1)\right)_{\frac{\alpha}{2}}$$

For $\lambda_L(y_0)$ we get

$$\frac{\alpha}{2} = P(Y \geq y_0) = P(Z \leq z) = P(2\lambda Z \leq 2\lambda z) = P\left(\chi^2(2y_0) \leq 2\lambda n\right) = \lambda_L(y_0) = \frac{1}{2n}\chi^2(2y_0)_{\frac{\alpha}{2}}$$