Exercise 2

Problem 1

Let the probability density function (pdf) of a gamma distributed random variable X be given by:

$$f\left(x|\alpha,\beta\right)=\begin{cases}\dfrac{1}{\Gamma(\alpha)\beta^{\alpha}}x^{\alpha-1}e^{-\frac{x}{\beta}}, & x>0,\ \alpha>0,\ \beta>0\\ 0, & \text{elsewhere}\end{cases},$$

where $\Gamma(\alpha)$ is the gamma function. We write that $X\sim\Gamma(\alpha,\beta)$.

a) What is the distribution of cX, where c is a constant? Explain why $E[X]=c\alpha\beta$ and why the variance of X is given by $c^2\alpha\beta^2$.

b) Show that $\alpha=\dfrac{p}{2}$ and $c=\dfrac{2}{\beta}$ gives a chi squared distribution with p degrees of freedom. Assume Y is chi squared distributed with p degrees of freedom. What is the distribution of bY where b is a constant?

c) Let $Z_1,Z_2,\ldots,Z_n$ be a random sample from a $N\left(\mu,\sigma^2\right)$ and let $S^2=\dfrac{1}{n-1}\sum_{i=1}^{n}\left(Z_i-\bar{Z}\right)^2$ be an estimator for the variance. Show that $S^2\sim\Gamma\left(\dfrac{n-1}{2},\dfrac{2\sigma^2}{n-1}\right)$. What is the variance of S^2?

d) Show that for a gamma distributed random variable, X, we have for $k>-\alpha$ that $E\left[X^k\right]=\dfrac{\Gamma(\alpha+k)\beta^k}{\Gamma(\alpha)}$. Let S be the estimator for σ. Show that

$$E[S]=\frac{\Gamma\left(\frac{n}{2}\right)2^{\frac{1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)\cdot(n-1)^{\frac{1}{2}}}\sigma.$$

e) Suggest an unbiased estimate for σ and find the variance of this estimator.

Problem 2 from the book.

3.20

3.23

3.30a