Chapter 1. Probability Theory

Sample space *S* - All possible outcomes of a particular experiment.

Event *A* – Subset of *S*

Probability – *P(A)*. $P(A): S \rightarrow \mathbb{R} \cap [0,1]$

σ - algebra (Definition 1.2.1)

A collection of subsets of *S, B,* that fulfills

1. $\phi \in B$
2. $A \in B \Rightarrow A^c \in B$
3. $A_1, A_2, \ldots \in B \Rightarrow \bigcup_{i=1}^{\infty} A_i \in B$

S finite or countable $\Rightarrow$ *B* is all subset of *S*

S not countable for instance. *S*=$(-\infty, \infty)$. B is all possible intervals of the type (a,b), (a, b], [a, b), [a,b]. (Borel *σ - algebraen*)

Probability function (Definition 1.2.4)

Given *S* and *B*, a probability function is a function that satisfies

1. $P(A) \geq 0 \;\forall\; A \in B$
2. $P(S) = 1$
3. $\left.\begin{matrix} A_1, A_2, \ldots \in B \\ A_i \cap A_j = \phi,\; i \neq j \end{matrix}\right\} \Rightarrow P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

Calculus of probability

1. Addition rule (1.2.9)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

2. Multiplication rule

$P(A \cap B) = P(A|B) \cdot P(B)$ (1.3.3)

3. The law of total probability (1.2.11)

$S = \bigcup_{i=1}^{\infty} C_i,\ C_i \cap C_j = \phi,\ \forall i \neq j$. Then

$$P(A) = \sum_{i=1}^{\infty} P(A \cap C_i) = \sum_{i=1}^{\infty} P(A|C_i) P(C_i).$$

4. Bayes rule (1.3.5)

$$P(C_i|A) = \frac{P(C_i \cap A)}{P(A)} = \frac{P(A|C_i)P(C_i)}{\sum_{j=1}^{\infty} P(A|C_j)P(C_j)}$$

Independence (1.3.12)

$$P(A \cap B) = P(A) \cdot P(B)$$

Random variables

X random variable. $X : S \to R$ (Definition 1.4.1)

Distribution function

$F_X(x) = P_X(X \le x),\ \forall x$ (Definition 1.5.1)

X is discrete if $F_X(x)$ is a step function

X is continuous if $F_X(x)$ is a continuous function

Probability mass function (X discrete)

$$f_X(x) = P_X(X = x) = P\left(\{s_j \in S : X(s_j) = x\}\right)$$

$$F_X(a) = \sum_{x \le a} P_X(X = x)$$

Support of X: All x for which $P_X(X = x) > 0$

Probability density function (X continuous**)**

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt,\ \forall x$$

$$f_X(x) = \frac{d}{dx} F_X(x)$$

Support of X: All x for which $f_X(x) > 0$

Identical distributed variables (Definition 1.5.8)

If $P(X \in A) = P(Y \in A) \forall A \in B$ then X and Y are identical distributed

Chapter 2. Transformations and Expectations

Distributions of Functions of a Random Variable (2.1)

X is defined on $\mathcal{X}$ og $Y = g(X)$ is defined on Υ.

$$P(Y \in A) = P(g(X) \in A) = P(\{x \in \mathcal{X} : g(x) \in A\}) = P\left(X \in g^{-1}(A)\right)$$

$$g^{-1}(A) = \{x \in \mathcal{X} : g(x) \in A\}$$

$$g^{-1}(y) = \{x \in \mathcal{X} : g(x) = y\}$$

X discrete

$$f_Y(y) = P(Y = y) = \sum_{x \in g^{-1}(y)} P(X = x), \text{ for y} \in \Upsilon.$$

X continous

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(\{x \in \mathcal{X} : g(x) \le y) = \int_{\{x \in \mathcal{X}:\, g(x) \le y\}} f_X(x) dx$$

Monotone transformations (page 50)

g increasing if $u > v \Rightarrow g(u) > g(v)$

g decreasing if $u > v \Rightarrow g(u) < g(v)$

g increasing or decreasing $\Leftrightarrow$ g is monotone.

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} f_X\left(g^{-1}(y)\right)\left|\frac{d}{dy} g^{-1}(y)\right|, y \in \Upsilon \\ 0, \text{ elles} \end{cases}$$

Theorem 2.1.8

Let *X* have pdf $f_X(x)$, let $Y = g(X)$ and let χ be the sample space. Suppose there exist a partition, $A_0, A_1, \ldots, A_k$ of χ such that $P(X \in A_0) = 0$ and $f_X(x)$ is continuous on each A_i. Further suppose there exist functions $g_1(x), \ldots, g_k(x)$ defined on $A_1, \ldots, A_k$, repectively, satisfying:

i. $g(x) = g_i(x), \text{ for } x \in A_i$

ii. $g_i(x)$ is monotone on A_i

iii. The set $\Upsilon = \{y : y = g(x_i) \text{ for some } x \in A_i\}$ is the same for each $i = 1, 2, \ldots, k$,

iv. and $g_i^{-1}(y)$ has a continuous derivative on Υ, for each $i = 1, 2, \ldots, k$

Then $f_Y(y) = \begin{cases} \sum_{i=1}^{k} f_X\left(g_i^{-1}(y)\right)\left|\frac{d}{dy} g^{-1}(y)\right|, y \in \Upsilon \\ 0 \qquad \text{otherwise} \end{cases}$

Expected Value (2.2)

If $\begin{cases} \int_{-\infty}^{\infty} |x| f_X(x)dx < \infty \\ \sum_x |x| P(X = x) < \infty \end{cases}$ then $E[X]$ = $\begin{cases} \int_{-\infty}^{\infty} x f_X(x)dx < \infty \\ \sum_x xP(X = x) < \infty \end{cases}$

Definition 2.2.1

$$E[g(X)] = \begin{cases} \int_{-\infty}^{\infty} g(x) f_X(x)dx \\ \sum_{x \in \mathrm{X}} g(x)P(X = x) \end{cases}$$

$$E\left[\sum_{i=1}^{n} g(X_i)\right] = \sum_{i=1}^{n} E[g(X_i)]$$

Momentgenerating function (2.3**)**

$$M_X(t) = E\left[e^{tX}\right] = \begin{cases} \int_{-\infty}^{\infty} e^{tx} f_X(x)dx, \ \text{X continuous} \\ \sum_x e^{tx} P(X = x), \ \text{X discrete} \end{cases}$$

$$M_X^n(t) = E\left[X^n e^{tX}\right]$$