Problem 1

Assume $X_1,\ldots,X_n$ are iid Poisson distributed with parameter λ, i. e. the pmf for each of the variables is given by:

$$f\left(x|\lambda\right)=P\left(X=x\right)=\frac{\lambda^x e^{-\lambda}}{x!},\ x=0,1,2,\ldots$$

where $\lambda\in\left(0,\infty\right)$.

a) Show that $f\left(x|\lambda\right)$ is a member of an exponential family of distributions. Derive the maximum likelihood estimator for λ, $\hat{\lambda}$, based on $X_1,\ldots,X_n$.

b) Derive the score statistic based on $X_1,\ldots,X_n$. Is $\hat{\lambda}$ a uniform minimum variance unbiased estimator (UMVUE) for λ? Explain your answer.

c) It is of interest to estimate the probability $P\left(X=0\right)=e^{-\lambda}$. What is the maximum likelihood estimator for the probability $P\left(X=0\right)$? Find the moment generating function for $Y=\sum_{i=1}^{n}X_i$ and use it to find the expected value of the maximum likelihood estimator for $P\left(X=0\right)$. Is it unbiased?

d) Suppose there exist an unbiased estimator $W\left(X_1,\ldots,X_n\right)$ for $\tau\left(\lambda\right)=e^{-\lambda}$. Derive the Cramer-Rao lower bound of the variance of $W\left(X_1,\ldots,X_n\right)$. Can $W\left(X_1,\ldots,X_n\right)$ attain this lower bound?

e) Suppose U and V are independent Poisson distributed with parameters λ_1 and λ_2 respectively. Let $Z=U+V$. Show that the conditional distribution of U given $Z=z$ is binomial$\left(z,\frac{\lambda_1}{\lambda_1+\lambda_2}\right)$. What is the conditional distribution of X_1 given that $Y=\sum_{i=1}^{n}x_i$?

f) Let the estimator $\hat{\tau}\left(X_1,\ldots X_n\right)$ be defined by $\hat{\tau}\left(X_1,\ldots,X_n\right)=\begin{cases}1 & \text{if } X_1=0\\ 0, & \text{otherwise}\end{cases}$.

Show that $\hat{\tau}\left(X_1,\ldots X_n\right)$ is an unbiased estimator for $\tau\left(\lambda\right)=e^{-\lambda}$. Explain also why the estimator $E\left(\hat{\tau}\left(X_1,\ldots X_n\right)\middle|Y=\sum_{i=1}^{n}X_i\right)$ is a unique best unbiased estimator of $\tau\left(\lambda\right)=e^{-\lambda}$.

g) Show that $E\left(\hat{\tau}\left(X_1,\dots X_n\right) \middle| Y = \sum_{i=1}^{n} X_i \right) = \left(1-\frac{1}{n}\right)^{\sum_{i=1}^{n} X_i}$ and find the variance of this estimator. Compare the variance to its lower bound.

Problem 2

Another way to get information about λ is to register the time between events in a Poisson process. Let $T_1, T_2, \dots, T_n$ be iid and represent n such times, each with a pdf given by

$$f_T\left(t|\lambda\right) = \begin{cases} \lambda e^{-\lambda t}, t > 0 \\ 0, \text{ otherwise} \end{cases}.$$

a) Show that $U = \sum_{i=1}^{n} T_i$ is a sufficient statistic for λ and verify if it is minimal or not. What is the distribution of $U = \sum_{i=1}^{n} T_i$?

b) Show that $\left[\frac{\chi^2_{2n,1-\frac{\alpha}{2}}}{2\sum_{i=1}^{n} T_i}, \frac{\chi^2_{2n,\frac{\alpha}{2}}}{2\sum_{i=1}^{n} T_i} \right]$ is a $1-\alpha$ confidence interval for λ. What is the expected length of this interval? (Hint. You can use that if V is gamma distributed with parameters α and β, then $E\left[V^k\right] = \frac{\Gamma\left(\alpha+k\right)\beta^k}{\Gamma\left(\alpha\right)}$, $k > -\alpha$.)

c) Explain why $\sqrt{n}\left(\overline{T}_n - \frac{1}{\lambda}\right) \xrightarrow{D} N\left(0, \frac{1}{\lambda^2}\right)$ where $\overline{T}_n = \frac{1}{n}\sum_{i=1}^{n} T_i$ and $\xrightarrow{D}$ means convergence in distribution. Use this to construct a $1-\alpha$ confidence interval for λ based on $\overline{T}_n$, using its asymptotic distribution. Compare the expected length of this interval to the one in b) when $\alpha = 0.05$ and n=20. What is the asymptotic distribution of $\overline{T}_n^{-1}$?