Løysingsforslag TMA 4295 H-18

Oppgåve 1

a)

$$f(x|\lambda) = \frac{1}{x!} e^{-\lambda} \lambda^x = \frac{1}{x!} e^{-\lambda} e^{x \ln \lambda} = h(x)\, c(\lambda)\, e^{x\, w(\lambda)}, \quad x = 0, 1, 2, \ldots$$

$\lambda > 0 \Rightarrow f(x|\lambda)$ is an exponential family.

$$f(\underline{x}|\lambda) = \frac{\lambda^{\sum x_i} e^{-n\lambda}}{\prod x_i!} = L(\lambda|\underline{x})$$

$$\Rightarrow \ell(\lambda|\underline{x}) = \sum x_i \ln \lambda - n\lambda - \ln\left(\prod x_i!\right)$$

$$\frac{d\ell}{d\lambda} = \frac{\sum x_i}{\lambda} - n = 0 \Rightarrow \lambda = \frac{\sum x_i}{n}$$

$$\frac{d^2\ell}{d\lambda^2} = -\frac{\sum x_i}{\lambda^2} < 0 \Rightarrow L(\lambda|\underline{x}) \text{ has a maksimum}$$

for $\lambda = \bar{x} \Rightarrow \quad MLE = \hat{\lambda} = \bar{X}$

b)

$$S(X_1, \ldots, X_n|\lambda) = \frac{d\ell}{d\lambda} = \frac{\sum X_i}{\lambda} - n$$

$$E[\hat{\lambda}] = E\left[\frac{\sum X_i}{n}\right] = \frac{n\lambda}{n} = \lambda$$

$$Var[\hat{\lambda}] = \frac{1}{n^2} Var\left[\sum X_i\right] = \frac{n\lambda}{n^2} = \frac{\lambda}{n}$$

$$Var\left[S(\underline{X}|\lambda)\right] = \frac{1}{\lambda^2} Var\left[\sum_{i=1}^{n} X_i\right] = \frac{1}{\lambda^2} \cdot n\lambda = \frac{n}{\lambda}$$

$\Rightarrow$ C-R lower bound $= \frac{\lambda}{n} \Rightarrow \hat{\lambda}$ ~~= UMVUE~~ is a UMVUE estimator.

c)

The invariance property $\Rightarrow$ $MLE = e^{-\hat{\lambda}}$

$Y = \sum X_i \sim$ Poisson$(m\lambda) \Rightarrow M_Y(t) = \sum_{y=0}^{\infty} \frac{e^{ty}(m\lambda)^y e^{-m\lambda}}{y!} = e^{-m\lambda} \sum_{y=0}^{\infty} \frac{(m\lambda e^t)^y}{y!}$

$= e^{-m\lambda} \cdot e^{m\lambda e^t} = e^{m\lambda(e^t - 1)}$

$\hat{\lambda} = \frac{Y}{m} \Rightarrow E[e^{-\hat{\lambda}}] = M_Y(-\frac{1}{m}) = e^{m\lambda(e^{-\frac{1}{m}} - 1)}$

$= e^{m\lambda(1 - \frac{1}{m} + \frac{1}{2m^2} - \cdots - 1)} = e^{-\lambda} \cdot e^{\frac{1}{2m^2} - \cdots} \neq e^{-\lambda} \Rightarrow$ MLE

estimator is biased.

d) $E[W(X_1, \ldots, X_m)] = e^{-\lambda} = \tau(\lambda) \Rightarrow \tau'(\lambda) = -e^{-\lambda}$

$Var[S(\underline{X}|\lambda)] = \frac{m}{\lambda} \Rightarrow$ C-R lower bound is $e^{-2\lambda} \cdot \frac{\lambda}{m}$

$S(\underline{X}|\lambda) = \frac{m}{\lambda}\left(\frac{\sum X_i}{m} - \lambda\right)$ is a linear function of $\hat{\lambda}$

but cannot be written as $a(\lambda)(W(X_1, \ldots, X_m) - e^{-\lambda}) \Rightarrow$ the C-R lower bound cannot be attained.

e) $P(U=u \mid Z=z) = \dfrac{P(U=u \cap V=z-u)}{P(Z=z)}$

$= \dfrac{\frac{\lambda_1^u e^{-\lambda_1}}{u!} \cdot \frac{\lambda_2^{z-u} e^{-\lambda_2}}{(z-u)!}}{\frac{(\lambda_1+\lambda_2)^z e^{-\lambda_1-\lambda_2}}{z!}} = \frac{z!}{u!(z-u)!} \cdot \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^u \cdot \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{z-u}$

$= \binom{z}{u}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^u\left(1 - \frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{z-u} \Rightarrow \sim B(z, \frac{\lambda_1}{\lambda_1+\lambda_2})$

With $\lambda_2 = (m-1)\lambda$ and $\lambda_1 = \lambda$ we get

$X_1 \mid \sum_{i=1}^{m} X_i = \sum_{i=1}^{m} x_i \sim B(\sum x_i, \frac{1}{m})$

f) $E[\hat{\tau}(X_1, \ldots, X_m)] = 1 \cdot P(X_1 = 0) = e^{-\lambda} \Rightarrow \hat{\tau}$ unbiased

$f(x|\lambda)$ is an exponential family$^{\lambda \in (0,\infty)}$ $\Rightarrow Y = \sum_{i=1}^{m} X_i$ is a

Complete statistic for λ

Hence $E[\hat{\tau}(X_1, \ldots, X_m) \mid Y = \sum_{i=1}^{m} X_i]$ is a unique best unbiased

estimator of $\tau(\lambda) = e^{-\lambda}$

g) $X_1 \mid \sum X_i = \sum x_i \sim B(\sum x_i, \frac{1}{m})$

$\Rightarrow E[\hat{\tau}(X_1, \ldots, X_m) \mid \sum X_i = \sum x_i] = 1 \cdot P(X_1 = 0 \mid Y = \sum x_i)$

$= \binom{\sum x_i}{0} \left(\frac{1}{m}\right)^0 \left(1 - \frac{1}{m}\right)^{\sum x_i} = \left(1 - \frac{1}{m}\right)^{\sum x_i}$

$\Rightarrow E[\hat{\tau}(X_1, \ldots, X_m) \mid Y = \sum X_i] = \left(1 - \frac{1}{m}\right)^{\sum_{i=1}^{m} X_i}$

$$Var\left[\left(1 - \frac{1}{m}\right)^{\sum X_i}\right] = E\left[\left(1 - \frac{1}{m}\right)^{2 \sum X_i}\right] - \left(e^{-\lambda}\right)^2$$

$$= \sum_{y=0}^{\infty} \left(1 - \frac{1}{m}\right)^{2y} \frac{(m\lambda)^y e^{-m\lambda}}{y!} - e^{-2\lambda} = \underbrace{\sum_{y=0}^{\infty} \frac{\left(m\lambda - 2\lambda + \frac{\lambda}{m}\right)^y e^{-\left(m\lambda - 2\lambda + \frac{\lambda}{m}\right)}}{y!}}_{1} \cdot e^{-2\lambda + \frac{\lambda}{m}} - e^{-2\lambda}$$

$$= e^{-2\lambda + \frac{\lambda}{m}} - e^{-2\lambda} = e^{-2\lambda}\left(e^{\frac{\lambda}{m}} - 1\right) = e^{-2\lambda}\left(1 + \frac{\lambda}{m} + \frac{\lambda^2}{2m^2} + \ldots - 1\right)$$

$$= \underbrace{e^{-2\lambda} \cdot \frac{\lambda}{m}}_{\text{C-R lower bound}} + e^{-2\lambda}\left(\frac{\lambda^2}{2m^2} + \ldots\right) > e^{-2\lambda} \cdot \frac{\lambda}{m}$$

Can use momentgenerating function for the variance.

Problem 2

a) $f(\underline{t}|\lambda) = \lambda^m e^{-\lambda \sum_1^m t_i} = e^{-\lambda \sum t_i + m \ln \lambda} = g(\sum t_i, \lambda) \cdot \underset{h(x)}{\underset{\shortparallel}{1}}$

$\Rightarrow \; U = \sum_{i=1}^m T_i$ is sufficient.

$$\frac{f(\underline{x}|\lambda)}{f(\underline{y}|\lambda)} = \frac{e^{-\lambda \sum x_i + m \ln \lambda}}{e^{-\lambda \sum y_i + m \ln \lambda}} = e^{-\lambda(\sum x_i - \sum y_i)} \quad \text{independent of } \lambda$$

$\Leftrightarrow \sum x_i = \sum y_i \Rightarrow \quad U = \sum_1^m T_i$ is minimal sufficient.

$T_i \sim \exp(\lambda) \sim \text{gamma}(1, \frac{1}{\lambda})$, $\quad T_1, \dots, T_m$ iid

$\Rightarrow \sum_{i=1}^m T_i \sim \text{gamma}(m, \frac{1}{\lambda})$

b) $2\lambda \sum_{i=1}^m T_i \sim \text{gamma}(m, 2) \sim \chi^2(2m)$

Hence $P(\chi^2_{2m, 1-\frac{\alpha}{2}} \le 2\lambda \sum_{i=1}^m T_i \le \chi^2_{2m, \frac{\alpha}{2}}) = 1-\alpha$

$$\Rightarrow P\left[\frac{\chi^2_{2m, 1-\frac{\alpha}{2}}}{2\sum T_i} \le \lambda \le \frac{\chi^2_{2m, \frac{\alpha}{2}}}{2\sum T_i}\right] = 1-\alpha$$

$$L = \left[\chi^2_{2m, \frac{\alpha}{\lambda}} - \chi^2_{2m, 1-\frac{\alpha}{2}}\right] \cdot \frac{1}{2\sum T_i} \Rightarrow E[L] = \frac{1}{2}\left[\chi^2_{2m, \frac{\alpha}{2}} - \chi^2_{2m, 1-\frac{\alpha}{2}}\right] E\left[\frac{1}{\sum T_i}\right]$$

$$E\left[\left(\sum T_i\right)^{-1}\right] = \frac{\Gamma(m-1)}{\Gamma(m)} \cdot \left(\frac{1}{\lambda}\right)^{-1} = \frac{\lambda}{m-1}$$

$$\Rightarrow E[L] = \frac{\lambda}{2(m-1)}\left[\chi^2_{2m, \frac{\alpha}{2}} - \chi^2_{2m, 1-\frac{\alpha}{2}}\right]$$

5

c) $E[T_i] = \frac{1}{\lambda}$ and $\bar{T}_m$ is the MLE estimator of $\frac{1}{\lambda} = \tau(\lambda)$

$S(T) = \frac{d}{d\lambda}\left(\ln\lambda - \lambda T\right) = \frac{1}{\lambda} - T \Rightarrow I(\lambda) = Var(T) = \frac{1}{\lambda^2}$

$\Rightarrow$ C-R lower bound $= \frac{(\tau'(\lambda))^2}{\frac{1}{\lambda^2}} = \frac{\left(-\frac{1}{\lambda^2}\right)^2}{\frac{1}{\lambda^2}} = \frac{1}{\lambda^2}$

$\bar{T}_m$ is (asymptotic or MLE estimator efficient) $\Rightarrow \sqrt{m}\,(\bar{T}_m - \frac{1}{\lambda}) \xrightarrow{D} N(0, \frac{1}{\lambda^2})$

Can also use the central limit theorem.

$P\left(-z_{\frac{\alpha}{2}} \le \frac{\bar{T}_m - \frac{1}{\lambda}}{\frac{1}{\lambda\sqrt{m}}} \le z_{\frac{\alpha}{2}}\right) = 1-\alpha \Leftrightarrow P\left(-z_{\frac{\alpha}{2}} \le \frac{\bar{T}_m\lambda - 1}{\frac{1}{\sqrt{m}}} \le z_{\frac{\alpha}{2}}\right) = 1-\alpha$

$\Leftrightarrow P\left(\frac{1 - z_{\frac{\alpha}{2}}\frac{1}{\sqrt{m}}}{\bar{T}_m} \le \lambda \le \frac{1 + z_{\frac{\alpha}{2}}\cdot\frac{1}{\sqrt{m}}}{\bar{T}_m}\right) = 1-\alpha$

$L_{asymp} = \frac{1}{\bar{T}_m}\left[1 + z_{\frac{\alpha}{2}}\cdot\frac{1}{\sqrt{m}} - 1 + z_{\frac{\alpha}{2}}\frac{1}{\sqrt{m}}\right] = \frac{\frac{2}{\sqrt{m}}\cdot z_{\frac{\alpha}{2}}}{\bar{T}_m}$

$E[L_{asymp}] = 2\sqrt{m}\, z_{\frac{\alpha}{2}}\cdot E\left[\frac{1}{\Sigma T_i}\right] = \frac{\lambda}{m-1}\, 2\sqrt{m}\cdot z_{\frac{\alpha}{2}}$

$\frac{E[L]}{E[L_{asymp}]} = \frac{\chi^2_{2m,\frac{\alpha}{2}} - \chi^2_{2m,1-\frac{\alpha}{2}}}{4\sqrt{m}\cdot z_{\frac{\alpha}{2}}} = 0.985^- = \frac{L}{L_{asymp}}$

$\Rightarrow$ asymptotic interval is a little longer.

$g(\bar{T}_m) = \frac{1}{\bar{T}_m}$. Let $\theta = E[\bar{T}_m] = \frac{1}{\lambda} \Rightarrow g(\theta) = \frac{1}{\theta}$ and $g'(\theta) = -\frac{1}{\theta^2}$

$= -\lambda^2 \Rightarrow |g'(\theta)|^2 = \lambda^4$

From the Delta theorem we get.

$\sqrt{m}\left(\bar{T}_m^{-1} - g(\theta)\right) = \sqrt{m}\left(\bar{T}_m^{-1} - \lambda\right) \xrightarrow{D} N\left(0, \frac{1}{\lambda^2}\cdot\lambda^4\right) \sim N(0, \lambda^2)$