TMA4295 Statistical inference
Exercise 2 - solution

Problem 1
$X \sim \Gamma(\alpha, \beta)$

a) To find the distribution of cX we can use theorem 2.1.5.

$$y = cx \Rightarrow x = \frac{y}{c} \Rightarrow \frac{dx}{dy} = \frac{1}{c}$$

$$\Rightarrow f_Y(y) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\frac{y^{\alpha-1}}{c^{\alpha-1}}e^{-\frac{y}{c\beta}}\frac{1}{c} = \frac{1}{\Gamma(\alpha)(c\beta)^\alpha}y^{\alpha-1}e^{-\frac{y}{c\beta}} \sim \Gamma(\alpha, c\beta)$$

$E(Y) = cE(Y) = c\alpha\beta$
$Var(Y) = c^2Var(X) = c^2\alpha\beta^2$.

b) $\alpha = \frac{p}{2}$ and $c = \frac{2}{\beta}$

$$\Rightarrow f_Y(y) = \frac{1}{\Gamma\left(\frac{p}{2}\right)}\frac{1}{2^{p/2}}y^{p/2-1}e^{-\frac{y}{2}} \sim \chi^2_p$$

since Y it also a $\Gamma(\frac{p}{2}, 2)$ then from part (a) we have that $bY \sim \Gamma\left(\frac{p}{2}, 2b\right)$

c) $Z_1, Z_2, ..., Z_n \sim N(\mu, \sigma^2)$ and S^2 variance estimator,
Recall that

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

and from the from the previous point we also know that $\frac{(n-1)S^2}{\sigma^2} \sim \Gamma\left(\frac{n-1}{2}, 2b\right)$

$$\Rightarrow S^2 \sim \Gamma\left(\frac{n-1}{2}, \frac{2\sigma^2}{n-1}\right)$$

$Var(S^2) = \frac{n-1}{2}\frac{4\sigma^4}{(n-1)^2} = \frac{2\sigma^4}{n-1}$

d) $X \sim \Gamma(\alpha, \beta)$ and $k > -\alpha$

$$E(X^k) = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha+k-1}e^{-\frac{x}{\beta}}dx = \frac{\beta^k}{\Gamma(\alpha)}\int_0^\infty \frac{x^{\alpha+k-1}}{\beta^{\alpha+k}}e^{-\frac{x}{\beta}}dx = \frac{\beta^k}{\Gamma(\alpha)}\Gamma(\alpha+k).$$

To compute $E(S)$ we can use the previous part with $k = 1/2$ since S^2 has a gamma distribution. Hence

$$E(S) = \frac{\Gamma(\frac{n-1}{2} + \frac{1}{2})\left(\frac{2\sigma^2}{n-1}\right)^{1/2}}{\Gamma(\frac{n-1}{2})} = \frac{\Gamma(\frac{n}{2})2^{1/2}\sigma}{\Gamma(\frac{n-1}{2})(n-1)^{1/2}}$$

e) We can then choose as unbiased estimator

$$\hat{S} = \frac{\Gamma(\frac{n-1}{2})(n-1)^{1/2}}{\Gamma(\frac{n}{2})2^{1/2}}S.$$

$$Var(\hat{S}) = \left(\frac{\Gamma(\frac{n-1}{2})(n-1)^{1/2}}{\Gamma(\frac{n}{2})2^{1/2}}\right)^2 Var(S) = \left(\frac{\Gamma(\frac{n-1}{2})(n-1)^{1/2}}{\Gamma(\frac{n}{2})2^{1/2}}\right)^2 (E(S^2) - E(S)^2) =$$

$$= \left(\frac{\Gamma(\frac{n-1}{2})(n-1)^{1/2}}{\Gamma(\frac{n}{2})2^{1/2}}\right)^2 (\sigma^2 - (\sigma/c)^2) = \sigma^2\left(\frac{\Gamma(\frac{n-1}{2})^2(n-1)}{\Gamma(\frac{n}{2})2} - 1\right)$$

Problem 3.20
X random variable with the pdf $f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}$

a) Mean:

$$E(x) = \int_0^\infty \frac{2}{\sqrt{2\pi}} e^{-x^2/2} x dx = -\frac{2}{\sqrt{2\pi}} e^{-x^2/2}\Big|_0^\infty = \frac{2}{\sqrt{2\pi}}.$$

Variance: since $Var(X) = E(x^2) - E(x)^2$ we need to compute $E(x^2)$.

$$E(x^2) = \int_0^\infty \frac{2}{\sqrt{2\pi}} e^{-x^2/2} x^2 dx = -\frac{2}{\sqrt{2\pi}} e^{-x^2/2} x\Big|_0^\infty + \int_0^\infty \frac{2}{\sqrt{2\pi}} e^{-x^2/2} dx = 1$$

$$\Rightarrow Var(x) = 1 - \frac{2}{\pi}.$$

b) We notice using the transformation $y = x^2$ and so $x = \sqrt{y}$ that:

$$f_Y(y) = \frac{2}{\sqrt{2\pi}} \frac{1}{2\sqrt{y}} e^{-y/2} = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{2}} y^{-\frac{1}{2}} e^{-y/2} = \frac{1}{\Gamma(\frac{1}{2}) 2^{\frac{1}{2}}} y^{\frac{1}{2}-1} e^{(-y/2)}$$

that is gamma distributed with $\alpha = \frac{1}{2}$ and $\beta = 2$.

Problem 3.23
The Pareto distribution has pdf :

$$f(x) = \frac{\beta\alpha^\beta}{x^{\beta+1}} \quad \alpha < x < \beta, \quad \alpha > 0, \quad \beta > 0.$$

a) Verify that $f(x)$ is a pdf:

$$\int_\alpha^\infty \frac{\beta\alpha^\beta}{x^{\beta+1}} dx = \beta\alpha^\beta \left[-\frac{1}{\beta} x^{-\beta} \right]_\alpha^\infty = 1.$$

b) Mean and variance:

$$E(x) = \int_\alpha^\infty \frac{\beta\alpha^\beta}{x^{\beta+1}} x dx = \frac{\beta\alpha^\beta}{(\beta-1)\alpha^{\beta-1}} = \frac{\beta\alpha}{\beta-1}$$

$$E(x^2) = \int_\alpha^\infty \frac{\beta\alpha^\beta}{x^{\beta+1}} x^2 dx = \frac{\beta\alpha^\beta}{(\beta-2)\alpha^{\beta-2}} = \frac{\beta\alpha^2}{\beta-2}$$

$$\Rightarrow Var(x) = \frac{\beta\alpha^2}{\beta-2} - \left(\frac{\beta\alpha}{\beta-1} \right)^2.$$

c) $E(x^2)$ does not exist for $\beta < 2 \Rightarrow$ the variance does not exist.

Problem 3.30

a) Let's rewrite the pdf of a binomial random variable

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x} (1-p)^n e^{x \log(\frac{p}{1-p})}$$

that is an exponential family with

$$c(p) = (1-p)^n, \quad t(x) = x, \quad \omega(p) = \log(\frac{p}{1-p}), \quad h(x) = \binom{n}{x}$$

$$\omega'(p) = \frac{1}{p(1-p)} \qquad \omega''(p) = \frac{2p-1}{p^2(1-p)^2}$$

$$\frac{d}{dp} \log c(p) = -\frac{n}{1-p} \qquad \frac{d^2}{dp^2} \log c(p) = -\frac{n}{(1-p)^2}$$

Theorem 3.4.2 implies that

$$Var\left(\frac{1}{p(1-p)}X\right) = \frac{n}{(1-p)^2} - E\left(\frac{X(2p-1)}{p^2(1-p)^2}\right)$$

$$\Rightarrow \frac{1}{p^2(1-p)^2} Var(X) = \frac{n}{(1-p)^2} - \frac{np(2p-1)}{p^2(1-p)^2}$$

$$\Rightarrow Var(x) = np^2 - 2np^2 + np = np(1-p).$$