# TMA4295 Statistical inference
# Exercise 4 - solution

**Problem 4.15**
$X \sim Poisson(\theta)$ and $Y \sim Poisson(\lambda)$
$Z = X + Y \sim Poisson(\theta + \lambda)$

$$P(X = x|Z = z) = \frac{P(X = x, Y = y)}{P(Z = z)} = \frac{\frac{\theta^x e^{-\theta}}{x!} \frac{\lambda^{z-x} e^{-\lambda}}{(z-x)!}}{\frac{(\theta+\lambda)^z e^{-(\theta+\lambda)}}{z!}} = \binom{z}{x} \left(\frac{\theta}{\theta + \lambda}\right)^x \left(1 - \frac{\theta}{\theta + \lambda}\right)^{z-x}$$

which is a binomial with $p = \left(\frac{\theta}{\theta+\lambda}\right)$.

**Problem 4.30**
$Y|X = x \sim N(x, x^2)$ and $X \sim Uniform(0, 1)$.

**a)**

$$E(Y) = E(E(Y|X)) = E(X) = \frac{1}{2},$$

$$Var(Y) = E(Var(Y|X)) + Var(E(Y|X)) = \frac{1}{12} + \frac{1}{3} = \frac{5}{12},$$

$$Cov(X, Y) = E(XY) - E(X)E(Y) = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}.$$

**b)** Consider the transformation $U = \frac{Y}{X}$ and $V = X$, then we have $x = v$ and $y = uv$ and the Jacobian is $v$. Since $f_{x,y}(x, y) = \frac{1}{2x\sqrt{\pi}} exp(\frac{-(y-x)^2}{2x^2}) I_{(0,1)(x)}$ then

$$f_{U,V}(u, v) = \frac{1}{2v\sqrt{\pi}} e^{-\frac{(uv-u)^2}{2u^2}} I_{(0,1)}(v)v = g(u)h(v).$$

**Problem 4.31**
$Y|X = x \sim B(n, x)$ and $X \sim Uniform(0, 1)$.

**a)**

$$E(Y) = E(E(Y|X)) = E(nX) = \frac{n}{2},$$

$$Var(Y) = E(Var(Y|X)) + Var(E(Y|X)) = E(nX(1 - X)) + Var(nX) = \frac{n}{6} + \frac{n^2}{12}.$$

**b)**

$$f(x, y) = f(y|x)f(x) = \binom{n}{y} x^y (1 - x)^{n-y} I_{(0,1)}.$$

**c)**

$$f_Y(y) = \int_0^1 f(x, y)dx = \int_0^1 \binom{n}{y} x^y (1 - x)^{n-y} dx = \binom{n}{y} \frac{\Gamma(y + 1)\Gamma(n - y + 1)}{\Gamma(n + 2)}.$$

**Problem 4.32**
$Y|\Lambda \sim Poisson(\Lambda)$ and $\Lambda \sim Gamma(\alpha, \beta)$.

**a)**

$$f_Y(y) = \int_0^\infty f(y|\lambda)f(\lambda)d\lambda = \int_0^\infty \frac{\lambda^y e^{-\lambda}}{y!} \frac{1}{\Gamma(\alpha)\beta^\alpha}\lambda^{\alpha-1}e^{-\frac{\lambda}{\beta}}d\lambda$$

$$= \frac{1}{y!\Gamma(\alpha)\beta^\alpha}\int_0^\infty \lambda^{y+\alpha-1}exp\left(\frac{-\lambda}{\frac{\beta}{1+\beta}}\right)d\lambda = \frac{1}{y!\Gamma(\alpha)\beta^\alpha}\Gamma(y+\alpha)\left(\frac{\beta}{1+\beta}\right)^{y+\alpha}.$$

If $\alpha$ is an integer

$$f_Y(y) = \binom{y+\alpha-1}{y}\left(\frac{\beta}{1+\beta}\right)^y\left(\frac{1}{1+\beta}\right)^\alpha$$

that is a negative binomial.

$$E(Y) = E(E(Y|\Lambda)) = E(\Lambda) = \alpha\beta,$$

$$Var(Y) = Var(E(Y|\Lambda)) + E(Var(y|\Lambda)) = E(\Lambda) + Var(\Lambda) = \alpha\beta + \alpha\beta^2 = \alpha\beta(\beta+1).$$

**b)** $Y|N \sim Binomial(N,p)$, $N|\Lambda \sim Poisson(\Lambda)$ and $\Lambda \sim gamma(\alpha,\beta)$ from example 4.4.2 we see that

$$P(Y=y|\lambda) = \sum_{n=y}^\infty P(Y=y|N=n,\lambda)P(N=n|\lambda) = \frac{(p\lambda)^y e^{-p\lambda}}{y!},$$

so we have $Y|\Lambda \sim Poisson(p\Lambda)$, calculations similar to those in a) prove that $Y$ is negative binomial distributed (if $\alpha$ is a positive integer).

**Problem 4.35**

$X|P \sim Binomial(n,P)$ and $P \sim Beta(\alpha,\beta)$

**a)**

$$Var(X) = E(Var(X|P)) + Var(E(X|P)) = E(nP(1-P)) + Var(nP)$$

$$= n(E(P) - E(P^2)) + n^2Var(P) = nE(P) - nVar(P) - n(E(P)^2) + n^2Var(P)$$

$$= nE(P)(1-E(P)) + n(n-1)Var(P).$$

**b)**

$$Var(Y) = \alpha\beta + \alpha\beta^2 = \mu + \frac{\mu^2}{\alpha}.$$

**Problem 4.36**

$X_i|P_i \sim Bernoulli(P_i) \quad i = 1,...,k$

$P_i \sim Beta(\alpha,\beta)$.

We first compute

$$E(X_i) = E(E(X_i|P_i)) = E(P_i) = \frac{\alpha}{\alpha+\beta},$$

$$Var(X_i) = E(Var(X_i|P_i)) + Var(E(X_i|P_i)) = \frac{\alpha\beta}{(\alpha+\beta)^2}.$$

**a)** Since $Y = \sum_{i=1}^k X_i$

$$E(X) = \sum_{i=1}^k E(X_i) = \frac{n\alpha}{\alpha+\beta}.$$

**b)**

$$Var(X) = \sum_{i=1}^{k} Var(X_i) = \frac{n\alpha\beta}{(\alpha+\beta)^2}.$$

To find the distribution of Y let's consider the mgf

$$M_Y(t) = E(e^{\sum_i x_i t}) = E(e^{x_1 t})^n = E(E(e^{x_1 t}|p_1))^n.$$

Now

$$E(e^{x_1 t}|p_1) = \sum_{x_1=0}^{1} e^{x_1 t}\left(p_1^{x_1}(1-p)^{1-x_1}\right) = 1 - p + pe^t$$

$$\Rightarrow M_Y(t) = E(1-p+pe^t)^n = \left(1 - \frac{\alpha}{\alpha+\beta} + e^t\frac{\alpha}{\alpha+\beta}\right).$$

Which is the mgf of a *binomial*$(n, \frac{\alpha}{\alpha+\beta})$.

**c)** $X_i|P_i \sim Binomial(n_i, P_i)$ and $P_i \sim Beta(\alpha, \beta)$ for $i = 1, ..., k$.

$$E(X_i) = E(E(X_i|P_i)) = E(n_i P_i) = \frac{n_i\alpha}{\alpha+\beta},$$

$$Var(X_i) = E(Var(X_i|P_i)) + Var(E(X_i|P_i)) = \frac{n_i\alpha\beta(\alpha+\beta+n_i)}{(\alpha+\beta)^2(\alpha+\beta+1)},$$

$$\Rightarrow E(Y) = \sum_{i=1}^{k} E(X_i) = \frac{\alpha}{\alpha+\beta}\sum_{i=1}^{k} n_i$$

$$Var(Y) = \sum_{i=1}^{k} Var(X_i).$$

**Problem 4.58**

**a)**

$$Cov(X,Y) = \int\int (x-\mu_x)(y-\mu_y)f(x,y)dxdy$$
$$= \int (x-\mu_x)\int (y-\mu_y)f(y|x)dyf(x)dx$$
$$= \int (x-\mu_x)(E(Y|X)-\mu_y)f(x)dx$$
$$= Cov(X, E(Y|X)).$$

**b)**

$$Cov(X,(Y-E(Y|X))) = E((X-\mu_x)(Y-E(Y|X)))$$
$$= \int\int (x-\mu_x)(y-E(Y|x))f(y|x)dyf(x)dy$$
$$= \int (x-\mu_x)\int (y-E(Y|x))f(y|x)dyf(x)dx$$
$$= \int (x-\mu_x)(E(Y|x)-E(Y|x))f(x)dx = 0.$$

**c)**

$$
\begin{aligned}
Var(Y - E(Y|X)) &= \int\int (y - E(Y|x))^2 f(x,y) dx dy \\
&= \int\int (y - E(Y|x))^2 f(y|x) dy f(x) dx \\
&= \int Var(Y|x) f(x) dx = E(Var(Y|X)).
\end{aligned}
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