Exercise 8. TMA4295

Problem 1

Let $X \sim \text{gamma}(\alpha, \beta)$ where $\alpha, \beta > 0$ (see the density on page 624 in book). Show that

$$E(\ln X) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \ln \beta.$$

Hint. You can use the fact that the pdfs in the gamma distribution belong to an exponential famaily.

Problem 2

Let $X_1, X_2, \ldots$ be a sequence of independent random variables each with probability density function given by

$$f(x|\nu) = \frac{1}{\Gamma\left(\frac{\nu}{2}\right)2^{\frac{\nu}{2}}} x^{\frac{\nu}{2}-1} e^{\frac{-x}{2}}, \quad 0 < x < \infty, \quad \nu = 1$$

which is a chi square distribution with one degree of freedom.

a) Explain why $X_1, X_2, \ldots$ also can be considered as gamma distributed random variables with parameters α and β. Find α and β and use them to show that the expectation and variance of X_i equal 1 and 2 respectively, $i = 1,2,\ldots$.

b) Let $Z_n = \sum_{i=1}^{n} X_i$, n=1,2,… . What is the distribution of Z_n? Explain why $\sqrt{n}\left(\frac{Z_n}{n\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$ has a limiting standard normal distribution.

c) Let $W_n = \left(\frac{Z_n}{n\sqrt{2}}\right)^{\frac{1}{2}}$. Show that $\sqrt{n}\left(W_n - \left(\frac{1}{2}\right)^{\frac{1}{4}}\right) \to N\left(0, 2^{-3/2}\right)$ in distribution. Let

$Y_i,\ i = 1,2,\ldots,n$ be a sequence of independent and identical normally distributed variables with mean μ and variance σ^2. For μ known, an estimator for the standard deviation, σ, is given by $S_n = \sqrt{\frac{\sum_{i=1}^{n}(Y_i - \mu)^2}{n}}$. Find an approximation of the variance of S_n that applies for large n.

Problem 3

Let $X_1 \ldots X_n$ be iid uniformly distributed on the interval $[0,\theta]$. It was shown in exercise 7 that the statistic

$$T(\boldsymbol{X}) = \max\{X_1, \ldots, X_n\}$$

is sufficient for θ.

a) Find the moment estimator of θ. Can this be written as a function of $T(\boldsymbol{X})$? Give a comment.

b) Let $n = 3$ and assume that the observations are 0.1, 0.9, 8.0. Compute the moment estimate. Is this estimate of θ reasonable?

c) Derive the MLE for θ. Find its expectation, variance and Mean Squared Error (MSE), i.e. $E(\hat{\theta} - \theta)^2)$.

d) Find an unbiased estimator for θ on the form const$\cdot T(\boldsymbol{X})$. Find the estimator's variance and compare with the MSE of, respectively, the moment estimator and the MLE.

Problem 4 (Bayes estimation)

Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be n observations from $N(\theta, \sigma^2)$, where σ^2 is known. Assume that we have given a prior distribution for θ given by $N(m, \tau^2)$.

a) Show that the posterior distribution for θ given $\boldsymbol{X} = \boldsymbol{x}$ is given by

$$N\left(\frac{\sigma^2}{\sigma^2 + n\tau^2} m + \frac{n\tau^2}{\sigma^2 + n\tau^2} \bar{x}, \frac{\sigma^2 \tau^2}{\sigma^2 + n\tau^2}\right)$$

Try to minimize the computation (it may be cumbersome otherwise!) See also Example 7.2.16 in the book.

b) Which family of distributions is hence conjugate to the normal distribution?

c) What is the Bayes-estimator for θ? Which is the weight it puts on the prior knowledge versus the information from the data $\boldsymbol{X}$? Give a comment.