TMA4295 Statistical inference
Exercise 1 - solution

2.33

$M_X(t) = \mathrm{E}\left(e^{tX}\right)$, $\mathrm{E}\left(X^n\right) = \left.\frac{d^n M_X(t)}{dt^n}\right|_{t=0}$

a) Use the fact that $e^y = \sum_{k=0}^{\infty} \frac{y^k}{k!}$ for the computation of the moment generating function.
$\mathrm{E}(X) = \lambda$
$\mathrm{E}\left(X^2\right) = \lambda^2 + \lambda$
$\mathrm{Var}(X) = \lambda$

c) Use completing the square

$$\begin{aligned} x^2 - 2\mu x - 2\sigma^2 tx + \mu^2 &= x^2 - 2(\mu + \sigma^2 t)x \pm (\mu + \sigma^2 t)^2 + \mu^2 \\ &= (x - (\mu + \sigma^2 t))^2 - (2\mu\sigma^2 t + (\sigma^2 t)^2) \end{aligned}$$

and the fact that integrals of the probability density functions over the probability space are equal to 1 (in this case it leads to the normal distribution) in the computation of the moment generating function.
$\mathrm{E}(X) = \mu$
$\mathrm{E}\left(X^2\right) = \mu^2 + \sigma^2$
$\mathrm{Var}(X) = \sigma^2$

2.35

a) Use the fact that $x^r = e^{r\log(x)}$ and the substitution $y = \log(x)$ and completing the square together with the form of the normal distribution as in the exercise 2.33c).

b) Use the same transformation $x^r = e^{r\log(x)}$ and substitution $y = \log(x) - r$. The resulting integral is an odd function so the negative integral cancels the positive one.

2.38

a) Use the fact that $\sum_{x=0}^{\infty} \binom{r+x-1}{x} \left((1-p)e^t\right)^x \left(1-(1-p)e^t\right)^r = 1$ for $(1-p)e^t < 1$, since this is just sum of the pmf of the negative binomial distribution.
$\mathrm{E}\left(e^{tX}\right) = \left(\frac{p}{1-(1-p)e^t}\right)^r$, $t < -\log(1-p)$

b) Use the fact, that $M_{2pX}(t) = M_X(2pt)$. The limit can be computed with use of the L'Hospital rule and the limiting moment generating function is the moment generating function of the χ^2 squared distribution with $2r$ degrees of freedom (see tables).

3.28

Exponential family: $f(x|\boldsymbol{\theta}) = h(x)c(\boldsymbol{\theta})e^{\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x))}$

a) μ known: $h(x) = 1$, $c(\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}$, $w_1(\sigma^2) = -\frac{1}{2\sigma^2}$, $t_1(x) = (x-\mu)^2$
σ^2 known: $h(x) = e^{-\frac{(x)^2}{2\sigma^2}}$, $c(\mu) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(\mu)^2}{2\sigma^2}}$, $w_1(\mu) = \mu$, $t_1(x) = \frac{x}{\sigma^2}$

b) α known: $h(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}$, $c(\beta) = \frac{1}{\beta^\alpha}$, $w_1(\beta) = \frac{1}{\beta}$, $t_1(x) = -x$
β known: $h(x) = e^{-\frac{x}{\beta}}$, $c(\alpha) = \frac{1}{\Gamma\alpha\beta^\alpha}$, $w_1(\alpha) = \alpha - 1$, $t_1(x) = \log(x)$
α, β unknown: $h(x) = 1$, $c(\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}$, $w_1(\alpha) = \alpha - 1$, $w_2(\beta) = -\frac{1}{\beta}$, $t_1(x) = \log(x)$, $t_2(x) = x$

d) $h(x) = \frac{1}{x!}$, $c(\theta) = e^{-\theta}$, $w_1(\theta) = \log(\theta)$, $t_1(x) = x$

3.39

The exercise can be solved for $\mu = 0$ and $\sigma^2 = 1$ and using the substitution $z = \frac{x-\mu}{\sigma}$ afterwards, since we are working with the location-scale family.

a) Since the pdf is symmetrical around 0, 0 must be median. Verifying this, write

$$\mathrm{P}\left(Z \geq 0\right) = \int_0^\infty \frac{1}{\pi} \frac{1}{1+z^2} dz = \frac{1}{\pi} \tan^{-1}(z) \Big|_0^\infty = \frac{1}{2}$$

b) $\mathrm{P}\left(Z \geq 1\right) = \frac{1}{4}$ which also holds for $\mathrm{P}\left(Z \leq -1\right)$ by symmetry.