Exercise 11 Compulsory

Problem 1

A sample $X_1, ..., X_n$ is taken from a gamma distribution with parameters θ and $1/\theta$:

$$X_1, ..., X_n \sim gamma\left(\theta, \frac{1}{\theta}\right)$$

i. e. pdf of X_i is

$$f(x;\theta) = \frac{\theta^\theta}{\Gamma(\theta)} x^{\theta-1} e^{-\theta x} I_{\{x>0\}}, \quad \theta > 0.$$

Find a one-dimensional sufficient statistic for θ.

Problem 2

Let $X_1, X_2, \ldots X_n$, $n > 1$ be independent and identically binomial (1, p) distributed, $0 < p < 1$. Let $Y = \sum_{i=1}^{n} X_i$

a) Show that the distribution of Y is an exponential family and also that Y is a sufficient statistics for p.

b) Is Y also a minimal sufficient statistics for p? Explain also why Y is a complete statistics.

c) Determine $E\left(X_1 \middle| Y = \sum_{i=1}^{n} X_i\right)$ and its expectation. What do you know about the properties of the estimator $E\left(X_1 \middle| Y = \sum_{i=1}^{n} X_i\right)$?

Problem 3

Let $X_1, ..., X_n$ be a sample taken from a normal distribution with zero mean and

unknown variance θ^2:

$$X_1, ..., X_n \sim N(0, \theta^2)$$

a) Find the (expected) Fisher information.

b) Consider the following estimator of θ^2:

$$T_n = \frac{2}{n}X_1^2 + \frac{n-2}{n(n-1)}\sum_{i=2}^{n} X_i^2.$$

Is this estimator unbiased?

c) Is the estimator efficient? We call an estimator effcient if it attains its lower bound.

d) Find the MLE estimator of θ^2. Is it efficient?

Problem 4

Let X be Poisson(α) and Y be Poisson(β). To find out whether X, Y are identically distributed we will test

$$H_0 : \alpha = \beta \text{ vs } H_1 : \alpha \neq \beta$$

a) Write down the likelihood function $L(\alpha, \beta|x, y)$. Find the MLE for α, β in the full model and under H_0.

b) Show that the likelihood ration can be written

$$\lambda(x, y) = \left(\frac{x+y}{2x}\right)^x \left(\frac{x+y}{2y}\right)^y$$