Solution TMA 4295 2019

1a) $M_X(t) = \sum_{x=1}^{\infty} e^{tx}\theta(1-\theta)^{x-1} \overset{s=x-1}{=} \sum_{s=0}^{\infty} e^{t(s+1)}\theta(1-\theta)^s$

$= \theta e^t \sum_{s=0}^{\infty}(e^t(1-\theta))^s = \frac{\theta e^t}{1-e^t(1-\theta)}$ for $e^t(1-\theta)<1$ or $t<-\ln(1-\theta)$

$M_X'(t) = \frac{\theta e^t}{1-e^t(1-\theta)} + \frac{e^t(1-\theta)\theta e^t}{(1-e^t(1-\theta))^2} = \frac{\theta e^t}{(1-e^t(1-\theta))^2}$

$M_X'(0) = \frac{\theta}{\theta^2} = \frac{1}{\theta} = E[X]$

b) $L(\theta|\underline{x}) = \theta^n(1-\theta)^{\sum_{i=1}^n x_i - n}$

$\ln L(\theta|\underline{x}) = n\ln\theta + (\sum x_i - n)\ln(1-\theta)$

$\frac{d}{d\theta}\ln L(\theta|\underline{x}) = \frac{n}{\theta} - \frac{\sum x_i - n}{1-\theta} = 0 \iff n - n\theta - \theta\sum x_i + \theta n = 0$

$\Rightarrow \theta = \frac{n}{\sum x_i} = \frac{1}{\bar{x}}$

$\frac{d^2}{d\theta^2}\ln L(\theta|\underline{x}) = -\frac{n}{\theta^2} - \frac{\sum x_i - n}{(1-\theta)^2} < 0$ since $\sum_{i=1}^n x_i \geq n$

Hence $\hat{\theta}_{MLE} = \frac{1}{\bar{x}}$

$\mu(\theta) = \frac{1}{\theta} \Rightarrow \hat{\mu}_{MLE} = \frac{1}{\frac{1}{\bar{x}}} = \bar{x}$

$\sigma^2(\theta) = \frac{1-\theta}{\theta^2} \Rightarrow \hat{\sigma}^2_{MLE} = \frac{1-\frac{1}{\bar{x}}}{\frac{1}{\bar{x}^2}} = \bar{x}^2 - \bar{x}$

c) $S(\underline{X}|\theta) = \frac{d}{d\theta}\ln f(\underline{X}|\theta) = \frac{n}{\theta} - \frac{\sum X_i - n}{1-\theta} = -\frac{1}{1-\theta}\left(\sum X_i - \frac{n}{\theta}\right)$

$= -\frac{n}{1-\theta}\left(\bar{X} - \frac{1}{\theta}\right)$ which shows that $S(\underline{X}|\theta) = a(\theta)[W(\underline{X}) - \tau(\theta)]$

with $W(\underline{X}) = \bar{X}$ and $\tau(\theta) = \frac{1}{\theta} = E[\bar{X}]$. Hence $W(\underline{X})$ will attain its lower bound.

$$\frac{d}{d\theta} S(\underline{X}|\theta) = -\frac{n}{\theta^2} - \frac{\sum X_i - n}{(1-\theta)^2} \Rightarrow -E\left[S(\underline{X}|\theta)\right] = \frac{n}{\theta^2} + \frac{\frac{n}{\theta} - n}{(1-\theta)^2}$$

$$= \frac{n}{\theta^2} + \frac{n - n\theta}{\theta(1-\theta)^2} = \frac{n(1-\theta)}{\theta^2(1-\theta)^2} = \frac{n}{\theta^2(1-\theta)}$$

$$\tau(\theta) = \frac{1}{\theta} \Rightarrow \tau'(\theta) = -\frac{1}{\theta^2}$$

Hence $Var(\bar{X}) = \dfrac{\frac{1}{\theta^4}}{\frac{n}{\theta^2(1-\theta)}} = \underline{\dfrac{(1-\theta)}{n\theta^2}}$

d)

$$\left.\begin{array}{l} E[X_i] = \frac{1}{\theta}, \quad i = 1, 2, \ldots \\ Var[X_i] = \frac{1-\theta}{\theta^2} < \infty, \quad i = 1, 2, \ldots \end{array}\right\} \Rightarrow \frac{\sqrt{n}\left(\bar{X}_n - \frac{1}{\theta}\right)}{\sqrt{\frac{1-\theta}{\theta^2}}} \xrightarrow{D} N(0,1)$$

according to the central limit theorem. This can be written

as $\sqrt{n}\left(\bar{X}_n - \frac{1}{\theta}\right) \xrightarrow{D} N\left(0, \frac{1-\theta}{\theta^2}\right)$

Let $g(t) = \frac{1}{t} \Rightarrow g'(t) = -\frac{1}{t^2}$

$g(\bar{X}_n) = \frac{1}{\bar{X}_n}, \quad g\left(\frac{1}{\theta}\right) = \theta, \quad g'\left(\frac{1}{\theta}\right) = -\theta^2$

Using the delta theorem we get.

$$\sqrt{n}\left(\frac{1}{\bar{X}_n} - \theta\right) \xrightarrow{D} N\left(0, \frac{\theta^4(1-\theta)}{\theta^2}\right) \sim N(0, \theta^2(1-\theta))$$

Hence the asymptotic variance is $\theta^2(1-\theta)$.

e) $f(x|\theta) = \theta e^{(x-1)\ln(1-\theta)} = \frac{\theta}{1-\theta} e^{x\ln(1-\theta)}$

With $h(x) = 1$, $c(\theta) = \frac{\theta}{1-\theta}$, $t(x) = x$ and $w(\theta) = \ln(1-\theta)$

we get that $f(x|\theta)$ is a member of an exponential family.

Since $t(X) = X$, we have that $\sum_{i=1}^{n} X_i$ is a sufficient statistics.

$$\frac{f(\underline{x}\mid\theta)}{f(\underline{y}\mid\theta)} = \frac{\theta^n (1-\theta)^{\sum x_i - n}}{\theta^n (1-\theta)^{\sum y_i - n}} = (1-\theta)^{\sum x_i - \sum y} \text{ independent of}$$

$\theta \iff \sum x_i = \sum y_i \implies \sum X_i$ is a minimal sufficent statistic.

f) $E[\hat{\tau}(X_1,\dots,X_n)] = 1 \cdot P(X=1) + 0 \cdot P(X \neq 1) = 1 \cdot \theta = \theta$

Since $f(x\mid\theta)$ is a member of an exponential family and $\theta \in (0,1]$ we know that $T = \sum_{i=1}^{n} X_i$ is a complete statistic.

$$E[\hat{\tau}(X_1,\dots,X_n) \mid \sum_{i=1}^{n} X_i = t) = 1 \cdot P(X_1 = 1 \mid \sum_1^n X_i = t)$$

$$= \frac{P(X_1 = 1 \cap \sum_1^n X_i = t)}{P(\sum_1^n X_i = t)} = \frac{P(X_1 = 1 \cap \sum_2^n X_i = t-1)}{P(\sum_1^n X_i = t)}$$

$$= \frac{\theta \cdot \binom{t-2}{n-2} \theta^{n-1} (1-\theta)^{t-n}}{\binom{t-1}{n-1} \theta^n (1-\theta)^{t-n}} = \frac{\binom{t-2}{n-2}}{\binom{t-1}{n-1}} = \frac{n-1}{t-1}$$

this shows that $\text{UMVUE} = \dfrac{n-1}{\sum X_i - 1}$

2a) $L(\theta\mid\underline{x}) = \theta^n e^{-\theta \sum X_i} \implies \ln L(\theta\mid\underline{x}) = n \ln\theta - \theta \sum X_i$

$$\frac{d \ln L(\theta\mid\underline{x})}{d\theta} = \frac{n}{\theta} - \sum X_i = 0 \iff \theta = \frac{n}{\sum X_i} \text{ or } \hat{\theta}_{MLE} = \frac{1}{\bar{x}} \quad \left(\frac{d^2 \ln L(\theta\mid x)}{d\theta^2} = -\frac{n}{\theta^2} < 0 \right)$$

$$\lambda(\underline{x}) = \frac{\theta_0^n e^{-\theta_0 \sum X_i}}{\left(\frac{1}{\bar{x}}\right)^n e^{-\frac{1}{\bar{x}} \sum X_i}} = \frac{\theta_0^n \bar{x}^n e^{-\theta_0 \sum X_i}}{e^{-n}} = \frac{\theta_0^n \sum X_i^n e^{-\theta_0 \sum X_i}}{n^n e^{-n}}$$

$$= \left(\frac{\theta_0 e}{m}\right)^m y^m e^{-\theta_0 y}$$

$$\frac{d}{dy}\lambda(\underline{x}) = k\left(m y^{m-1} e^{-\theta_0 y} - \theta_0 e^{-\theta_0 y} y^m\right) = k e^{-\theta_0 y} y^{m-1}(m - \theta_0 y)$$

$\frac{d}{dy}\lambda(\underline{x}) \geq 0$ for $y \leq \frac{m}{\theta_0}$ and $\frac{d}{dy}\lambda(x) < 0$ for $y > \frac{m}{\theta_0}$

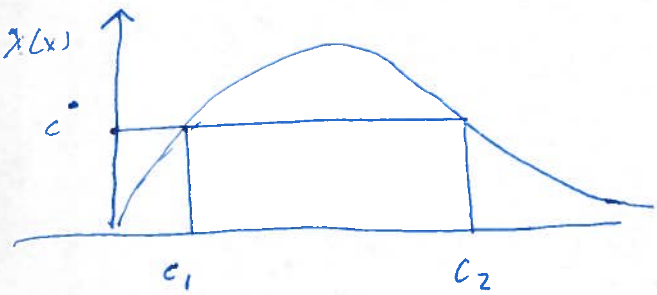
Hence $\lambda(x)$ has a maximum for $y = \frac{m}{\theta_0}$ and $\lambda(\underline{x}) = 0$ for $y = 0$

and $\to 0$ ~~for~~ as $y \to \infty$.

The rejection region is given by $\lambda(\underline{x}) \leq c \Leftrightarrow y^m e^{-\theta_0 y} \leq c\left(\frac{m}{\theta_0 e}\right)^m = c^*$

$\Leftrightarrow y = \sum x_i \leq c_1$ or $y = \sum x_i \geq c_2$

or $y \in \{(0, c_1] \cup [c_2, \infty)\}$



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a) $E[X|\theta] = \frac{1}{\theta}$. $E[X] = E\left[\frac{1}{\theta}\right] = \frac{\Gamma(\alpha-1)}{\Gamma(\alpha)} \cdot \beta^{-1} = \underline{\frac{1}{(\alpha-1)\beta}}$

$$Var[X] = E[Var(X|\theta)] + Var[E[X|\theta]] = E\left[\frac{1}{\theta^2}\right] + Var\left[\frac{1}{\theta}\right]$$

$$= E\left[\frac{1}{\theta^2}\right] + E\left[\frac{1}{\theta^2}\right] - \left(E\left[\frac{1}{\theta}\right]\right)^2 = \frac{2}{(\alpha-1)(\alpha-2)} \cdot \frac{1}{\beta^2} - \frac{1}{(\alpha-1)^2\beta^2} = \underline{\frac{\alpha}{\beta^2(\alpha-1)^2(\alpha-2)}}$$

b) $f(x|\theta) = \theta^m e^{-\theta \sum x_i}$

$$f(x|\theta)\cdot\pi(\theta) = \theta^m e^{-\theta\sum x_i} \frac{1}{\Gamma(\alpha)} \frac{\theta^{\alpha-1}}{\beta^\alpha} e^{-\frac{\theta}{\beta}}$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \theta^{\alpha+m-1} \cdot e^{-\left(\sum x_i + \frac{1}{\beta}\right)\theta} = \frac{\Gamma(\alpha+m)\cdot\beta^m}{\Gamma(\alpha)(\beta m\bar{x}+1)^{\alpha+m}} \cdot \underbrace{\frac{1}{\frac{\Gamma(\alpha+m)\cdot\beta^{\alpha+m}}{(\beta m\bar{x}+1)^{\alpha+m}}} \theta^{\alpha+m-1} e^{-\frac{\theta}{\left(\frac{\beta}{\beta m\bar{x}+1}\right)}}}_{\Gamma\left(\alpha+m, \frac{\beta}{\beta m\bar{x}+1}\right)}$$

$$\Rightarrow \pi(\theta|\underline{x}) = \frac{1}{\Gamma(\alpha+m)\cdot\left(\frac{\beta}{m\bar{x}\beta+1}\right)^{\alpha+m}} \theta^{\alpha+m-1} e^{-\frac{\theta}{\left(\frac{\beta}{m\bar{x}\beta+1}\right)}}$$

Thereby $\hat{\theta}_B = \dfrac{\alpha+m}{m\bar{x}\beta+1}$

c) $\hat{\theta}_B^{-1} = \dfrac{\Gamma(\alpha+m-1)\cdot\left(\frac{\beta}{m\bar{x}\beta+1}\right)^{-1}}{\Gamma(\alpha+m)} = \dfrac{m\bar{x}\beta+1}{\beta(\alpha+m-1)}$

$$\frac{2(m\bar{x}\beta+1)}{\beta}\theta \sim \chi^2(2(\alpha+m))$$

Hence $P\left(\chi^2_{2(\alpha+m),1-\frac{\alpha}{2}} \leq \dfrac{2(\Sigma x_i\beta+1)}{\beta}\cdot\theta \leq \chi^2_{2(\alpha+m),\frac{\alpha}{2}}\right) = 1-\alpha.$

$\Rightarrow P\left(\dfrac{\beta}{2(\Sigma x_i\beta+1)}\chi^2_{2(\alpha+m),1-\frac{\alpha}{2}} \leq \theta \leq \dfrac{\beta}{2(\Sigma x_i\beta+1)}\chi^2_{2(\alpha+m),\frac{\alpha}{2}}\right) = 1-\alpha.$

and a credible interval is $\left[\dfrac{\beta}{2(\Sigma x_i\beta+1)}\chi^2_{2(\alpha+m),1-\frac{\alpha}{2}}, \dfrac{\beta}{2(\Sigma x_i\beta+1)}\chi^2_{2(\alpha+m),\frac{\alpha}{2}}\right]$