Some useful results about the Score statistics

$$\frac{d}{d\theta}\left[log\ f\left(X|\theta\right)\right].$$

We have

$$\int_{-\infty}^{\infty} f\left(x|\theta\right)dx=1 \Rightarrow \int_{-\infty}^{\infty} \frac{d}{d\theta} f\left(x|\theta\right)dx = \int_{-\infty}^{\infty} \frac{\frac{d}{d\theta} f\left(x|\theta\right)}{f\left(x|\theta\right)} f\left(x|\theta\right)dx = 0$$

Thereby:

$$\int_{-\infty}^{\infty} \frac{d}{d\theta}\left[log\ f\left(x|\theta\right)\right] f\left(x|\theta\right)dx = 0$$

$$\Leftrightarrow E\left[\frac{d}{d\theta}\left[log\ f\left(X|\theta\right)\right]\right] = 0 \qquad (1)$$

Further:

$$\int_{-\infty}^{\infty} \frac{d}{d\theta}\left[\frac{d}{d\theta}\left[log\ f\left(x|\theta\right)\right] f\left(x|\theta\right)\right]dx = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{d^2}{d\theta^2} log\ f\left(x|\theta\right) f\left(x|\theta\right)dx + \int_{-\infty}^{\infty} \frac{d}{d\theta} f\left(x|\theta\right)\frac{d}{d\theta}\left[log\ f\left(x|\theta\right)\right]dx = 0$$

or

$$\int_{-\infty}^{\infty}\left(\frac{d}{d\theta} log\ f\left(x|\theta\right)\right)^2 f\left(x|\theta\right)dx = -\int_{-\infty}^{\infty} \frac{d^2}{d\theta^2} log\ f\left(x|\theta\right) f\left(x|\theta\right)dx$$

Which gives:

$$Var\left[\frac{d}{d\theta} log\ f\left(X|\theta\right)\right] = -E\left[\frac{d^2}{d\theta^2} log\ f\left(X|\theta\right)\right] \qquad (2)$$

Applied on exponential families of distributions

$$f\left(x|\boldsymbol{\theta}\right) = h\left(x\right)c\left(\boldsymbol{\theta}\right)e^{\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)}$$

$$log\ f\left(x|\boldsymbol{\theta}\right) = log\, h\left(x\right) + log\, c\left(\boldsymbol{\theta}\right) + \sum_{i=1}^{k} w_i\left(\boldsymbol{\theta}\right)t_i\left(x\right)$$

$$\frac{\partial}{\partial\theta_j} log\ f\left(x|\boldsymbol{\theta}\right) = \frac{\partial}{\partial\theta_j} log\, c\left(\boldsymbol{\theta}\right) + \sum_{i=1}^{k} \frac{\partial}{\partial\theta_j} w_i\left(\boldsymbol{\theta}\right) t_i\left(x\right)$$

(3)

Since $E\left[\frac{\partial}{\partial\theta_j} log\ f\left(X|\boldsymbol{\theta}\right)\right] = 0$ from (1), we get:

$$E\left[\sum_{i=1}^{k} \frac{\partial}{\partial\theta_j} w_i\left(\boldsymbol{\theta}\right) t_i\left(X\right)\right] = -\frac{\partial}{\partial\theta_j} log\, c\left(\boldsymbol{\theta}\right), j = 1,2,\ldots,k$$

Now from (3) and (2):

$$Var\left[\sum_{i=1}^{k} \frac{\partial}{\partial\theta_j} w_i\left(\boldsymbol{\theta}\right) t_i\left(X\right)\right] = Var\left[\frac{\partial}{\partial\theta_j} log\ f\left(X|\boldsymbol{\theta}\right)\right] = -E\left[\frac{\partial^2}{\partial\theta_j^2} log\ f\left(X|\boldsymbol{\theta}\right)\right], j = 1,2,\ldots,k$$

$$\frac{\partial^2}{\partial\theta_j^2} log\ f\left(x|\boldsymbol{\theta}\right) = \frac{\partial^2}{\partial\theta_j^2} log\, c\left(\boldsymbol{\theta}\right) + \sum_{i=1}^{k} \frac{\partial^2}{\partial\theta_j^2} w_i\left(\boldsymbol{\theta}\right) t_i\left(x\right)$$

Thereby:

$$Var\left[\sum_{i=1}^{k} \frac{\partial}{\partial\theta_j} w_i\left(\boldsymbol{\theta}\right) t_i\left(X\right)\right] = -\frac{\partial^2}{\partial\theta_j^2} log\, c\left(\boldsymbol{\theta}\right) - E\left[\sum_{i=1}^{k} \frac{\partial^2}{\partial\theta_j^2} w_i\left(\boldsymbol{\theta}\right) t_i\left(X\right)\right]$$