Repetition week 44

Score statistic

$$S(\boldsymbol{X}|\theta) = \frac{\partial}{\partial\theta}\log f(\boldsymbol{X}|\theta)$$

$$E\left[S(\boldsymbol{X}|\theta)\right] = 0$$

$$Var\left[S(\boldsymbol{X}|\theta)\right] = I_X(\theta) = -E\left[\frac{\partial}{\partial\theta}S(\boldsymbol{X}|\theta)\right] = -E\left[\frac{\partial^2}{\partial\theta^2}\log f(\boldsymbol{X}|\theta)\right]$$

Let $\tau(\theta) = E\left[W(\boldsymbol{X})\right]$

<u>Cramer-Rao</u>

$$Var\left[W(\boldsymbol{X})\right] \geq \frac{\left(\frac{\partial}{\partial\theta}\tau(\theta)\right)^2}{I_X(\theta)}$$

<u>Cramer-Rao iid</u>

$$Var\left[W(\boldsymbol{X})\right] \geq \frac{\left(\frac{\partial}{\partial\theta}\tau(\theta)\right)^2}{nI_X(\theta)}$$

<u>Equality</u>

If and only if $S(\boldsymbol{X}|\theta) = a(\theta)\left[W(\boldsymbol{X}) - \tau(\theta)\right]$

Cramer-Rao in the multiparameter case

$$\boldsymbol{\theta} = (\theta_1, \ldots \theta_k)^t$$

Define the Score function $\boldsymbol{S}(\boldsymbol{X}|\boldsymbol{\theta}) = \begin{bmatrix} \frac{\partial}{\partial \theta_1} \log f(\boldsymbol{x}|\boldsymbol{\theta}) \\ \vdots \\ \frac{\partial}{\partial \theta_k} \log f(\boldsymbol{x}|\boldsymbol{\theta}) \end{bmatrix} = \nabla \log f(\boldsymbol{x}|\boldsymbol{\theta})$

Define the Fisher information $I(\boldsymbol{\theta}) = Cov\left[\boldsymbol{S}(\boldsymbol{X}|\boldsymbol{\theta})\right]$

We have as in the univariate case that $E\left[\boldsymbol{S}(\boldsymbol{X}|\boldsymbol{\theta})\right] = \mathbf{0}$ and

$I(\boldsymbol{\theta}) = E\left[\boldsymbol{S}(\boldsymbol{X}|\boldsymbol{\theta})\boldsymbol{S}(\boldsymbol{X}|\boldsymbol{\theta})^T\right] = -E\left[H(\boldsymbol{X}|\boldsymbol{\theta})\right]$ where

$h_{ij} = \frac{\partial}{\partial \theta_i}\frac{\partial}{\partial \theta_j}\log f(\boldsymbol{x}|\boldsymbol{\theta})$.

If $\boldsymbol{W}(\boldsymbol{X})$ is an unbiased estimator for $\boldsymbol{\theta}$. Then $\boldsymbol{I}(\boldsymbol{\theta})^{-1}$ is taken as an approximation to $Cov\left[\boldsymbol{W}(\boldsymbol{X})\right]$

Let $\tau = \tau(\boldsymbol{\theta})$ be univariate and let $\nabla \tau(\boldsymbol{\theta}) = \begin{bmatrix} \frac{\partial}{\partial \theta_1}\tau(\boldsymbol{\theta}) \\ \vdots \\ \frac{\partial}{\partial \theta_k}\tau(\boldsymbol{\theta}) \end{bmatrix}$

Theorem. For an estimator $W(\boldsymbol{X})$ with $E[W(\boldsymbol{X})] = \tau$, we have under similar regularity conditions as in the univariate case that

$$Var[W(\boldsymbol{X})] \geq (\nabla \tau(\boldsymbol{\theta}))^T (I(\boldsymbol{\theta}))^{-1} (\nabla \tau(\boldsymbol{\theta})).$$

<u>Sufficiency and Unbiasedness</u>

W unbiased estimator of $\tau(\theta)$.

T a sufficient statistic $E[W|T] = \tau(\theta)$ and $Var[W|T] \leq Var[W]$, $\forall \theta$

T complete $\Rightarrow E[W|T]$ is the unique best unbiased estimator for $\tau(\theta)$

Definition 6.2.21

Let $f(t|\theta)$ be a family of pdfs/pmfs for a statistic $T(\boldsymbol{X})$. The family is complete if

$E_\theta[g(T)] = 0 \Rightarrow P_\theta(g(T) = 0) = 1$, for all θ.