Problem 1

Let the probability mass function (pmf) of a Bernoulli random variable X be given by:

$$f\left(x|\theta\right)=\theta^{x}\left(1-\theta\right)^{1-x},\ x=0,1.\quad \theta\in\left(0,1\right).$$

Assume we have a random sample $\boldsymbol{X}_n = X_1, X_2,\ldots, X_n$ from this distribution.

a) Derive the joint probability mass function for $X_1, X_2,\ldots, X_n$. Show that $Y=\sum_{i=1}^{n} X_i$ is a sufficient statistic for θ.

b) What is the distribution of Y? Explain your answer. Derive the maximum likelihood estimator for θ based on $X_1, X_2,\ldots, X_n$ and find its mean and variance.

We are interested in estimating $\delta\left(\theta\right)=\theta\left(1-\theta\right)$.

c) What is the maximum likelihood estimator for $\delta\left(\theta\right)$, $\hat{\delta}\left(\boldsymbol{X}_n\right)$? Derive the expected value of $\hat{\delta}\left(\boldsymbol{X}_n\right)$. Show that $\delta^*\left(\boldsymbol{X}_n\right)=\frac{\sum_{i=1}^{n}X_i}{n-1}\left(1-\frac{\sum_{i=1}^{n}X_i}{n}\right)$ is an unbiased estimator for $\delta\left(\theta\right)$.

d) Derive the moment generating function for Y given in 1a). Explain carefully (calculations are not necessary) how this can be used to find the variance of $\delta^*\left(\boldsymbol{X}_n\right)$.

e) Is $\delta^*\left(\boldsymbol{X}_n\right)$ the unique UMVU estimator for $\delta\left(\theta\right)$? Explain your answer. Find the Cramer-Rao lower bound for $\mathrm{Var}\left(\delta^*\left(\boldsymbol{X}_n\right)\right)$.

f) Explain why $\hat{\delta}\left(\boldsymbol{X}_n\right)$ converges in probability to $\delta\left(\theta\right)=\theta\left(1-\theta\right)$ as $n\to\infty$. Show also that $\delta^*\left(\boldsymbol{X}_n\right)$ converges in probability to $\hat{\delta}\left(\boldsymbol{X}_n\right)$ as $n\to\infty$.

g) Assume $\theta \neq \frac{1}{2}$. Find the asymptotic distribution of $\sqrt{n}\left(\delta^*(\boldsymbol{X}_n)-\delta(\theta)\right)$. Is $\delta^*(\boldsymbol{X}_n)$ asymptotic efficient? Find also the asymptotic distribution when $\theta = \frac{1}{2}$.

Problem 2

Let $X_1, X_2, \dots X_n$ be independent and identically distributed random variables, each with a probability density function, pdf, given by:

$$f\left(x|\lambda\right) = \lambda^2 x e^{-\lambda x}, \quad 0 < x < \infty, \quad \lambda > 0 .$$

a) The probability density function above belongs to a well-known family. What is the family and what are the parameters? Show that the maximum likelihood estimator for λ is $\hat{\lambda} = \dfrac{2n}{\sum_{i=1}^{n} X_i}$.

b) Find the expectation and the variance of $\hat{\lambda}$. Derive a $1-\alpha$ confidence interval for λ.

c) Assume now that a prior distribution for λ is:

$$\pi\left(\lambda\right) = \theta e^{-\theta\lambda}, \quad 0 < \lambda < \infty, \quad \theta > 0 .$$

Derive the Bayes estimator, $\hat{\lambda}_B$, for λ given $\left(X_1, X_2, \dots X_n\right) = \left(x_1, x_2, \dots x_n\right)$. Derive also a $1-\alpha$ credible interval (set) for λ.