3.4 Exponential families.

Assume $\quad X \sim N(\mu, \sigma^2)$

$$\Rightarrow \quad f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty, \ -\infty < \mu < \infty, \ \sigma > 0$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} e^{-\frac{\mu^2}{2\sigma^2}} \cdot e^{-\frac{x^2}{2\sigma^2} + \frac{2x\mu}{2\sigma^2}}$$

Define $\quad h(x) = \frac{1}{\sqrt{2\pi}}, \quad c(\mu,\sigma) = \frac{1}{\sigma} e^{-\frac{\mu^2}{2\sigma^2}}, \quad w_1(\mu,\sigma^2) = -\frac{1}{2\sigma^2}, \quad t_1(x) = x^2$

$$w_2(\mu,\sigma^2) = \frac{\mu}{\sigma^2}, \quad t_2(x) = x$$

$$\Rightarrow \quad f(x|\mu,\sigma^2) = h(x) \cdot c(\mu,\sigma^2) \, e^{\sum_{i=1}^{2} w_i(\mu,\sigma^2) t_i(x)}$$

A family of pdfs or pmfs is called an exponential family if it can be expressed as:

$$f(x|\underline{\theta}) = h(x)\, c(\underline{\theta})\, e^{\sum_{i=1}^{k} w_i(\underline{\theta}) t_i(x)}$$

where $h(x) \geq 0$, $t_1(x), \ldots t_k(x)$ are real valued functions of x that cannot depend on θ.

$c(\underline{\theta}) \geq 0$ and $w_1(\underline{\theta}), \ldots, w_k(\underline{\theta})$ are real valued functions of $\underline{\theta}$ that cannot depend on x.

Binomial distribution

$$f(x|p) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \ldots, n$$

$$f(x|p) = \binom{n}{x} \left(\frac{p}{1-p}\right)^x (1-p)^n = \binom{n}{x} (1-p)^n \, e^{x \log\left(\frac{p}{1-p}\right)}$$

$$h(x) = \begin{cases} \binom{n}{x}, & x = 0, 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases}, \quad c(p) = (1-p)^n, \quad w(p) = \log\left(\frac{p}{1-p}\right), \quad t(x) = x.$$

Uniform distribution

$$f(x|\theta) = \begin{cases} \frac{1}{\theta}, & 0 \le x \le \theta \\ 0 & \text{otherwise} \end{cases} \quad = \quad e^{-\log\theta} I_{[0,\theta]}(x)$$

$I_{[0,\theta]} = g(x,\theta) \Rightarrow f(x|\theta)$ is not a member of an exponential family, but.

Binomial, negative binomial, Poisson, normal, gamma and beta are.

Some useful results.

Suppose $f(x|\theta)$ is a pdf.

Then $\int_{-\infty}^{\infty} f(x|\theta)\,dx = 1$ and $\int_{-\infty}^{\infty} \frac{d}{d\theta} f(x|\theta)\,dx = 0$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\frac{d}{d\theta} f(x|\theta)}{f(x|\theta)} f(x|\theta)\,dx = 0 \Rightarrow \int_{-\infty}^{\infty} \frac{d}{d\theta}\left[\log f(x|\theta)\right] f(x|\theta)\,dx = 0$$

$$\Rightarrow E\Big[\underbrace{\frac{d}{d\theta} \log f(X|\theta)}_{\text{Score statistic}}\Big] = 0 \qquad (1)$$

Further

$$\int_{-\infty}^{\infty} \frac{d}{d\theta}\left[\frac{d}{d\theta} \log f(x|\theta)\, f(x|\theta)\right] dx = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \left(\frac{d^2}{d\theta^2} \log f(x|\theta)\right) f(x|\theta)\,dx + \int_{-\infty}^{\infty} \underbrace{\frac{d}{d\theta} f(x|\theta)}_{= \frac{d}{d\theta}\left[\log f(x|\theta)\right]\cdot f(x|\theta)} \frac{d}{d\theta} \log f(x|\theta)\,dx = 0$$

or $\int_{-\infty}^{\infty} \left(\frac{d}{d\theta} \log f(x|\theta)\right)^2 \cdot f(x|\theta)\,dx = - \int_{-\infty}^{\infty} \left(\frac{d^2}{d\theta^2} \log f(x|\theta)\right) \cdot f(x|\theta)\,dx$

which gives:

$$Var\Big[\underbrace{\frac{d}{d\theta}\log f(X|\theta)}_{\text{Score statistic.}}\Big] = -E\Big[\frac{d^2}{d\theta^2}\log f(X|\theta)\Big] \qquad (2)$$

Now assume

$$f(x|\underline{\theta}) = h(x)\,c(\underline{\theta})\,e^{\sum_{i=1}^{k} w_i(\underline{\theta})\,t_i(x)}$$

Then

$$\log f(x|\underline{\theta}) = \log h(x) + \log c(\underline{\theta}) + \sum_{i=1}^{k} w_i(\underline{\theta})\,t_i(x)$$

$$\frac{\partial}{\partial\theta_j}\log f(x|\underline{\theta}) = \frac{\partial}{\partial\theta_j}\log c(\underline{\theta}) + \sum_{i=1}^{k}\frac{\partial}{\partial\theta_j} w_i(\underline{\theta})\,t_i(x) \qquad (3)$$

From(1): $$0 = E\Big[\frac{\partial}{\partial\theta_j}\log f(X|\theta)\Big] = \frac{\partial}{\partial\theta_j}\log c(\underline{\theta}) + E\Big[\sum_{i=1}^{k}\frac{d}{d\theta_j} w_i(\underline{\theta})\,t_i(X)\Big]$$

$$\Rightarrow E\Big[\sum_{i=1}^{k}\frac{d}{d\theta_j} w_i(\underline{\theta})\,t_i(X)\Big] = -\frac{\partial}{\partial\theta_j}\log c(\underline{\theta}) \qquad 3.4.4$$

From (3): $$Var\Big[\frac{\partial}{\partial\theta_j}\log f(X|\underline{\theta})\Big] = Var\Big[\sum_{i=1}^{k}\frac{\partial}{\partial\theta_j} w_i(\underline{\theta})\,t_i(X)\Big] \overset{(2)}{=} -E\Big[\frac{\partial^2}{\partial\theta_j^2}\log f(X|\underline{\theta})\Big]$$

$$\frac{\partial^2}{\partial\theta_j^2}\log f(x|\underline{\theta}) = \frac{\partial^2}{\partial\theta_j^2}\log c(\underline{\theta}) + \sum_{i=1}^{k}\frac{\partial^2}{\partial\theta_j^2} w_i(\underline{\theta})\,t_i(x)$$

$$E\Big[\frac{\partial^2}{\partial\theta_j^2}\log f(X|\underline{\theta})\Big] = \frac{\partial^2}{\partial\theta_j^2}\log c(\underline{\theta}) + E\Big[\sum_{i=1}^{k}\frac{\partial^2}{\partial\theta_j^2} w_i(\underline{\theta})\,t_i(X)\Big]$$

Hence $$Var\Big[\sum_{i=1}^{k}\frac{\partial}{\partial\theta_j} w_i(\underline{\theta})\,t_i(X)\Big] = -\frac{\partial^2}{\partial\theta_j^2}\log c(\underline{\theta}) - E\Big[\sum_{i=1}^{k}\frac{\partial^2}{\partial\theta_j^2} w_i(\underline{\theta})\,t_i(X)\Big] \qquad 3.4.5$$

Example Binomial distribution.

$$f(x|p) = \binom{m}{x}(1-p)^m e^{x \log\left(\frac{p}{1-p}\right)}$$

$$c(p) = (1-p)^m, \quad w(p) = \log\left(\frac{p}{1-p}\right), \quad t(x) = x$$

$$w'(p) = \frac{1}{p(1-p)},$$

$$\log c(p) = m \log(1-p) \Rightarrow \frac{d}{dp} \log c(p) = \frac{-m}{1-p}$$

From 3.4.4

$$E\left[\frac{1}{p(1-p)} x\right] = \frac{m}{1-p} \Rightarrow E[x] = \frac{p(1-p)\cdot m}{1-p} = mp.$$

$$\frac{d^2}{dp^2} \log c(p) = \frac{-m}{(1-p)^2}, \quad \frac{d^2}{dp^2} w(p) = \frac{-(1-2p)}{(p(1-p))^2}$$

$$Var\left[\frac{1}{p(1-p)} x\right] = \frac{m}{(1-p)^2} + \frac{(1-2p)E[x]}{(p(1-p))^2} \Rightarrow Var[x] = mp^2 + (1-2p)mp$$

$$= mp - mp^2 = mp(1-p).$$

<u>Definition 3.4.7.</u>

A curved exponential family is a family of densities of the form $f(x|\underline{\theta}) = h(x)c(\underline{\theta}) e^{\sum_{i=1}^{k} w_i(\underline{\theta}) t_i(x)}$ for which the dimension of $\underline{\theta}$ is equal to $d < k$.

Example: $X \sim N(\mu, \mu^2)$.