Location and Scale distributions

Definition 3.5.5

Let $f(x)$ be any pdf. Then for any $-\infty < \mu < \infty$ and any $\sigma > 0$, the family of pdfs $\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$ is called a location-scale family with standard pdf $f(x)$. μ is the location parameter and σ is the scale parameter.

$X \sim f(x)$, let $Y = \mu + \sigma X \Rightarrow X = \frac{Y-\mu}{\sigma}$

Transformation formula : $f_Y(y) = \frac{1}{\sigma} f\left(\frac{y-\mu}{\sigma}\right)$

Convenient to let standard pdf have $E[X]=0$ and $Var[X]=1$

Example $X \sim N(0,1) \Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \; -\infty < x < \infty$

μ and σ unknown.

$Y = \mu + \sigma X \Rightarrow f_Y(y) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \; -\infty < x < \infty$

<u>Location-family</u>. Set $\sigma = 1$, $Y = \mu + X \Rightarrow f_Y(y) = f(y-\mu)$

Examples.

$X \sim N(0, \beta^2)$, β fixed $\Rightarrow Y = \mu + X \sim N(\mu, \beta^2)$.

$f(x) = e^{-x}, \; x > 0$, $\quad Y = \mu + X \Rightarrow f_Y(y) = \begin{cases} e^{-(y-\mu)}, & y - \mu \geq 0 \\ 0, & \text{otherwise} \end{cases}$

<u>Scale family</u>, Set $\mu = 0$, $Y = \sigma X$

$\Rightarrow f_Y(y) = \frac{1}{\sigma} f\left(\frac{y}{\sigma}\right)$

Examples $X \sim N(1, \beta^2) \Rightarrow Y = \sigma X \sim N(1, \sigma^2\beta^2)$ or $f_Y(y) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma\beta} e^{-\frac{(y-1)^2}{2\sigma^2\beta^2}}$

$X \sim gamma(\alpha, \beta)$, α fixed.

$Y = \sigma X \Rightarrow Y \sim gamma(\alpha, \sigma\beta)$

The location parameter can be the mean, the median or the mode.

Theorem 3.56

Let $f(\cdot)$ be any pdf. Let $-\infty < \mu < \infty$. Let $\sigma > 0$

Then X is a random variable with pdf $\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$

$\Leftrightarrow$ there exists a random variable Z with pdf $f(z)$ and $X = \sigma Z + \mu$.

$\Rightarrow$ Let $g(x) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$, $\int_{-\infty}^{\infty} g(x)\,dx = 1$

Define $Z = \frac{X-\mu}{\sigma} \Rightarrow X = \sigma Z + \mu$

The pdf of Z, $h_Z(z) = \frac{1}{\sigma} f\left(\frac{\sigma z + \mu - \mu}{\sigma}\right) \cdot \sigma = f(z)$

$\Leftarrow$

Pdf of $Z = f(z)$. Define $X = \sigma Z + \mu \Rightarrow Z = \frac{X-\mu}{\sigma}$

$\Rightarrow f_X(x) = f\left(\frac{x-\mu}{\sigma}\right) \cdot \frac{1}{\sigma}$.

3.6 Inequalities and Identities.

Theorem 3.6.1 (Markov or Chebychev)

Let X be a random variable and $g(x)$ nonnegative

Then for any $r > 0$

$$P(g(X) \geq r) \leq \frac{E[g(X)]}{r}$$

Proof.

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx \geq \int_{\{x: g(x) \geq r\}} g(x) f_X(x)\,dx$$

$$\geq r \int_{\{x; g(x) \geq r\}} f_X(x)\,dx = r\,P(g(X) \geq r)$$

Example.

Let $g(x) = \left(\frac{X-\mu}{\sigma}\right)^2$

Then $P\left(\left(\frac{X-\mu}{\sigma}\right)^2 \geq t^2\right) \leq \frac{1}{t^2} E\left[\left(\frac{X-\mu}{\sigma}\right)^2\right] = \frac{1}{t^2}$

$\Longleftrightarrow P(|X-\mu| \geq t\sigma) \leq \frac{1}{t^2}$

and $P(|X-\mu| \geq 2\sigma) \leq \frac{1}{2^2} = \frac{1}{4}$

Example 3.6.3

$Z \sim N(0,1)$. Then $P(|Z| \geq t) = \sqrt{\frac{2}{\pi}}\,\frac{e^{-\frac{t^2}{2}}}{t}$

Proof. $P(|Z| > t) = 2\,P(Z > t) = \frac{2}{\sqrt{2\pi}} \int_t^{\infty} e^{-\frac{x^2}{2}}\,dx$

$\leq \frac{2}{\sqrt{2\pi}} \int_t^{\infty} \frac{x}{t}\, e^{-\frac{x^2}{2}}\,dx$ since $\frac{x}{t} > 1$

$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{t} \left[-e^{-\frac{x^2}{2}}\right]_t^{\infty} = \sqrt{\frac{2}{\pi}}\,\frac{e^{-\frac{t^2}{2}}}{t} \Rightarrow P(|Z| > 2) \leq \sqrt{\frac{2}{\pi}}\,\frac{e^{-2}}{2} = 0.054$

Assume $t > 0$. Then $P(X \geq a) \leq e^{-at} M_X(t)$ provided

$M_X(t)$ exist.

Proof. Let $g(X) = e^{tX}$

$$P(e^{tX} \geq e^{ta}) \leq \frac{E[g(X)]}{e^{ta}} = e^{-at} M_X(t)$$

4. Multiple Random Variables.

Definition 4.2.5

Let (X, Y) be a bivariate random vector with joint pdf/pmf $f(x,y)$ and marginal pdfs/pmfs $f_X(x)$ and $f_Y(y)$. Then X and Y are independent if $f(x,y) = f(x) \cdot f(y)$, $\forall (x,y)$

Lemma 4.2.7

X and Y independent $\iff$ $f(x,y) = g(x)\, h(y)$, $\forall (x,y)$

Theorem 4.2.12

Let X and Y be independent and $M_X(t)$ and $M_Y(t)$ exist. Let $Z = X + Y$. Then $M_Z(t) = M_X(t) \cdot M_Y(t)$.

4.3 Bivariate Transformations

$(X, Y) \sim$ bivariate random vector. Let $U = g_1(X,Y)$ and $V = g_2(X,Y)$. What is the distribution of (U,V).

Let $B \in R^2$

Then $P((U,V) \in B) = P((X,Y) \in A)$ where

$$A = \{(x,y) : (g_1(x,y), g_2(x,y)) \in B\}$$

Continuous case

Assume $U = g_1(X,Y)$ and $V = g_2(X,Y)$ defines a one to one transformation from A to B i.e.

For each $(u,v) \in B$ there is only one $(x,y) \in A$ such that $(u,v) = (g_1(x,y), g_2(x,y))$. Then there is an inverse transformation such that $x = h_1(u,v)$ and $y = h_2(u,v)$

The Jacobian of the transformation is defined as:

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial h_1(u,v)}{\partial u} \cdot \frac{\partial h_2(u,v)}{\partial v} - \frac{\partial h_2(u,v)}{\partial u} \cdot \frac{\partial h_1(u,v)}{\partial v}$$

and $f_{U,V}(u,v) = f_{X,Y}(h_1(u,v), h_2(u,v)) \, |J|$

$X \sim N(0,1)$, $Y \sim N(0,1)$, X, Y independent

$$\begin{aligned} U &= X + Y = g_1(X,Y) \\ V &= X - Y = g_2(X,Y) \end{aligned} \implies \begin{cases} X = \frac{U+V}{2} = h_1(U,V) \\ Y = \frac{U-V}{2} = h_2(U,V) \end{cases}$$

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{x^2}{2}} \cdot e^{-\frac{y^2}{2}}, \quad \infty - x < \infty, \quad -\infty < y < \infty$$

$\implies A = R^2, \quad B = R^2$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$f_{U,V}(u,v) = f_{X,Y}(h_1(u,v), h_2(u,v))\,|J|$$

$$= \frac{1}{2\pi} e^{-\frac{\left(\frac{u+v}{2}\right)^2}{2}} \cdot e^{-\frac{\left(\frac{u-v}{2}\right)^2}{2}} \cdot \left|-\tfrac{1}{2}\right|$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sqrt{2}} e^{-\frac{u^2}{4}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2}} e^{-\frac{v^2}{4}} = g(u)\cdot g(v)$$

$\Rightarrow$ U and V are independent $\sim N(0, \sqrt{2}^2)$

Theorem 4.3.5

X and Y independent $\Rightarrow$ $U = g(X)$ and $V = h(Y)$ are independent.

Proof. For any $u \in R$ and $v \in R$, let

$A_u = \{x : g(x) \le u\}$ and $B_v = \{y : h(y) \le v\}$

$$F_{U,V}(u,v) = P(U \le u \cap V \le v) = P(X \in A_u \cap Y \in B_v)$$

$= P(X \in A_u)\cdot P(Y \in B_v)$ and $f_{U,V}(u,v) = \frac{\partial^2}{\partial u \partial v} F_{U,V}(u,v)$

$$= \frac{\partial}{\partial u} P(X \in A_u) \cdot \frac{\partial}{\partial v} P(Y \in B_u)$$

Example 4.3.6

X and Y independent $\sim N(0,1)$

$U = \frac{X}{Y}$, $V = |Y|$

Then (X,Y) and $(-X,-Y)$ gives the same U and V and