Monotone likelihood ratio (MLR)

Assume $X_1, \dots, X_n \sim N(\theta, \sigma^2)$, σ^2 known

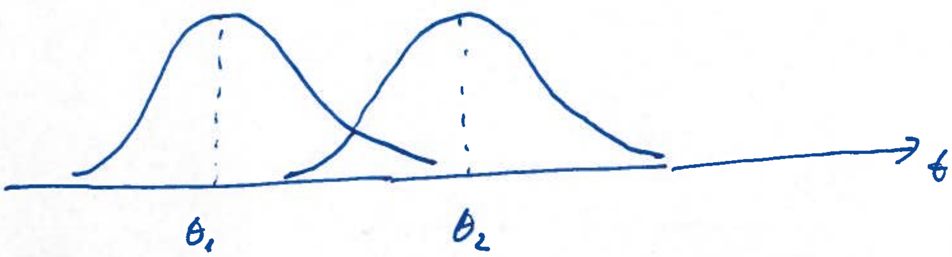
$T = \bar{X}$ is sufficient for θ with pdf/pmf $g(t|\theta)$

Let $\theta_2 > \theta_1$ and $H_0: \theta = \theta_1$ vs $H_1: \theta = \theta_2$.

$$\frac{g(\bar{x}|\theta_2)}{g(\bar{x}|\theta_1)} = \frac{k_1 \, e^{-\frac{n}{2\sigma^2}(\bar{x}-\theta_2)^2}}{k_1 \, e^{-\frac{n}{2\sigma^2}(\bar{x}-\theta_1)^2}} = e^{\frac{n\bar{x}}{\sigma^2}(\theta_2-\theta_1) - \frac{n}{2\sigma^2}(\theta_2^2-\theta_1^2)}$$

is a monotone increasing function in $\bar{x}$

$\Longleftrightarrow$ T has a MLR



What is the point with a MLR?

We have $\frac{d}{dt}\left[F(t|\theta_2) - F(t|\theta_1)\right] = \left[f(t|\theta_2) - f(t|\theta_1)\right]$

$$= f(t|\theta_1)\left[\frac{f(t|\theta_2)}{f(t|\theta_1)} - 1\right]$$

$\Downarrow$ monotone increasing.

$\Rightarrow$ $F(t|\theta_2) - F(t|\theta_1)$ has a minimum in the interior and maximum $= 0$ for $t = \pm\infty$

$\Rightarrow$ $F(t|\theta_2) \le F(t|\theta_1)$

If $\beta(\theta) = P(T > t_0)$ we have $\beta(\theta_2) \ge \beta(\theta_1)$ or nondecreasing.

<u>This leads to Karlin-Rubin 8.3.17</u>

$H_0: \theta \le \theta_0$ $\quad H_1: \theta > \theta_0$, T has a MLR

Let $\alpha = P(T > t_0|\theta_0)$, ~~a UMP test at level~~ If we ~~to~~ test rejects only if $T > t_0$, it is a UMP level-α test.

Chapter 9. Interval Estimation

Let $\underline{X} = X_1, \dots, X_n$ be a sample (often random) and let θ be a real valued distribution parameter.

Definition 9.1.1

An interval estimate θ is any pair of functions $L(x_1, \dots, x_n)$ and $U(x_1, \dots, x_n)$ of a sample that satisfies $L(\underline{x}) \le U(\underline{x})$, $\forall \underline{x} \in \mathcal{X}$. If $\underline{X} = \underline{x}$ is observed the inference $L(\underline{x}) \le \theta \le U(\underline{x})$ is made. The random interval $[L(\underline{X}), U(\underline{X})]$ is called an interval estimator.

Definition 9.1.4

The coverage probability for $[L(\underline{X}), U(\underline{X})]$ is the probability that $[L(\underline{X}), U(\underline{X})]$ covers θ, write $P(\theta \in [L(\underline{X}), U(\underline{X})])$
This probability may depend on θ.

Definition 9.1.5

The confidence coefficient of $[L(\underline{X}), U(\underline{X})]$ is the infimum of the coverage probabilities $\inf_{\theta} P(\theta \in [L(\underline{X}), U(\underline{X})])$

Example $X_1, \dots, X_n$ iid $N(\mu, \sigma^2)$, σ^2 known

$$P\left(-z_{\frac{\alpha}{2}} \le \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \le z_{\frac{\alpha}{2}}\right) = 1 - \alpha$$

$$\Leftrightarrow P\left(\bar{X} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right)$$

$L(\underline{X}) = \bar{X} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$, $U(\underline{X}) = \bar{X} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$

$L(\underline{X}) = \bar{X} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$, $U(\underline{X}) = \bar{X} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$

Confidence coefficient $= 1-\alpha$

9.2 Methods of finding Interval Estimators

9.2.1 Inverting a Test statistic.

$X_1, \dots, X_n$ iid $N(\mu, \sigma^2)$

$H_0: \mu = \mu_0$ $\quad H_1: \mu \neq \mu_0$ with test size α

Not reject if $|\bar{X} - \mu_0| \le z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$

$$\iff -z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \le \bar{X} - \mu_0 \le z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

$$\iff \mu_0 - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \le \bar{X} \le \mu_0 + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

Let $A(\mu_0) = \{(x_1, \dots, x_n) : \mu_0 - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \le \bar{x} \le \mu_0 + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\}$

$A(\mu_0)$ gives the sample values that are consistent with H_0 or the acceptance region.

On the other side

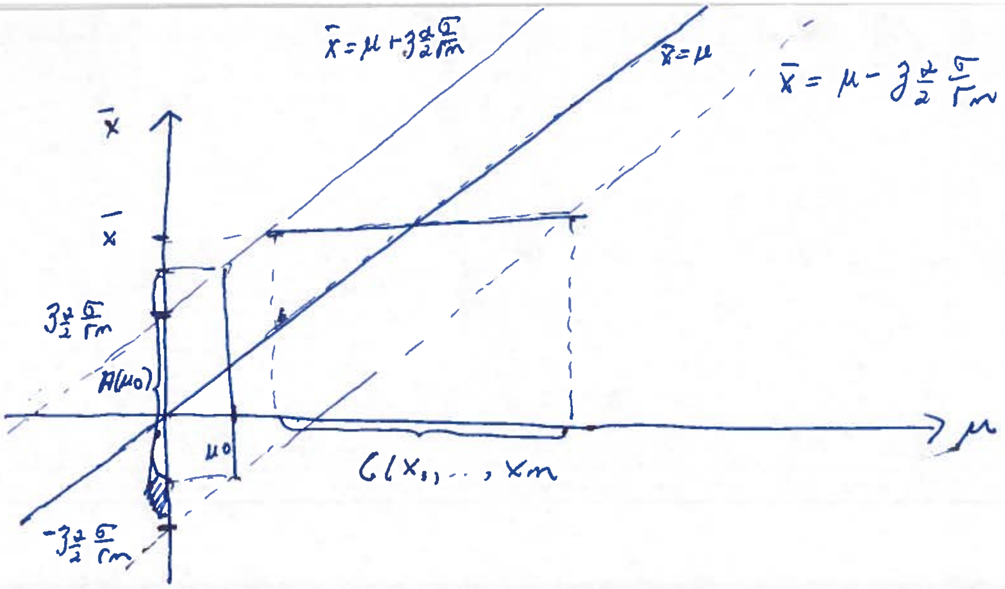
$$-z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \le \bar{x} - \mu_0 \le z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \iff \bar{x} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \le \mu_0 \le \bar{x} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

Define $C(x_1, \dots, x_n) = \{\mu_0 : \bar{x} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \le \mu_0 \le \bar{x} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\}$

It gives the parameter values that ~~make the~~ are ~~observations~~ plausible given $\bar{x}$ ~~under H_0~~ or the confidence interval

We have $(x_1, \dots, x_n) \in A(\mu_0) \iff \mu_0 \in C(x_1, \dots, x_n)$

Therefore $P(\underline{X} \in A(\mu_0)) = P(\mu_0 \in C(X_1, \dots, X_n))$



Theorem 9.2.2

For each $\theta_0 \in \Omega$ let $A(\theta_0)$ (all $\underline{x}$ values consistent with $\theta = \theta_0$) be the consistency region
acceptance region
for a level α-test. Define for each $\underline{x}$ $C(\underline{x})$ by $C(\underline{x}) = \{\theta_0 : \underline{x} \in A(\theta_0)\}$ (all θ_0 values for which $\underline{x}$ does not give rejection). Then $C(\underline{X})$ is a $1-\alpha$ confidence set.

Let $C(\underline{X})$ be a $1-\alpha$ confidence set. For any $\theta_0 \in \Omega$ define $A(\theta_0) = \{\underline{x}, \theta_0 \in C(\underline{x})\}$. Then $A(\theta_0)$ is the consistency region
acceptance region
of a level α-test of $H_0: \theta = \theta_0$.

Proof.

1. $P(\underline{X} \notin A(\theta_0) | \theta_0) \le \alpha \iff P(\underline{X} \in A(\theta_0) | \theta_0) \ge 1-\alpha$

This is true for any θ. Hence

$$P(\theta \in C(\underline{X})) = P(\underline{X} \in A(\theta) | \theta) \ge 1-\alpha$$

2. $P(\underline{X} \notin A(\theta_0)) = P(\theta_0 \notin C(\underline{X})) \le \alpha$.