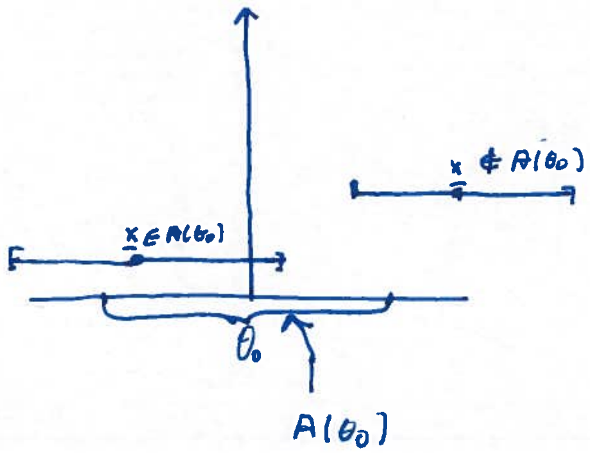
To 9.2.2



1. $\underline{x} \in A(\theta_0) \iff \theta_0 \in C(\underline{x})$. Given $H_0: \theta = \theta_0$, $x \in A(\theta_0)$ occurs more than $(1-\alpha)100\%$ of the time. Therefore $P(\theta_0 \in C(\underline{x}) | \theta_0) = P(\underline{X} \in A(\theta_0) | \theta_0) \geq 1-\alpha$. NB! The coverage probability is here the same for all θ.

2. For any θ_0, construct $A(\theta_0) = \{\underline{x} : \theta_0 \in C(\underline{x})\}$

Since the confidence intervals will cover θ_0 more than $(1-\alpha)100\%$ of the time, $\underline{x}$ will be ~~not~~ in the acceptance region for $H_0: \theta = \theta_0$ more than $(1-\alpha)100\%$ of the time and thereby $P(X \notin A(\theta_0) | \theta_0) = P(\theta_0 \notin C(\underline{X}) | \theta_0) \leq \alpha$.

Example 9.23 Inverting a LRT

$X_1, \dots, X_m$ iid $\exp(\frac{1}{\theta})$ i.e $f(x|\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$, $x > 0$, $\theta > 0$

$H_0: \theta = \theta_0$ vs $H_1: \theta \neq \theta_0$

$$f(x_1, \dots, x_m | \theta) = \frac{1}{\theta^m} e^{-\frac{1}{\theta}\sum_{i=1}^m x_i}, \quad x_i \in (0, \infty), \ i = 1, 2, \dots, m.$$

$$\lambda(\underline{x}) = \frac{\sup_{\theta=\theta_0} \frac{1}{\theta^m} e^{-\frac{1}{\theta}\sum x_i}}{\sup_{\theta} \frac{1}{\theta^m} e^{-\frac{1}{\theta}\sum x_i}} = \frac{\frac{1}{\theta_0^m} e^{-\frac{1}{\theta_0}\sum x_i}}{\frac{1}{\bar{x}^m} e^{-m}}$$

$$\bar{x} = \frac{\sum x_i}{m} \quad = \left(\frac{\sum x_i}{m\theta_0}\right)^m e^m e^{-\frac{1}{\theta_0}\sum x_i}.$$ For fixed θ_0 the

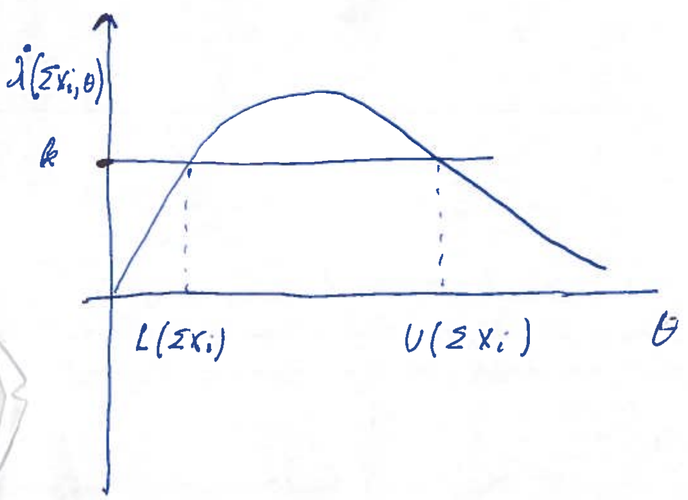
~~consistency~~ acceptance region is $A(\theta_0) = \{\underline{x} : \left(\frac{\sum x_i}{\theta_0}\right)^m e^{-\frac{\sum x_i}{\theta_0}} \geq k\}$

~~and for fixed~~ and for fixed $\underline{x}$

$$C(\underline{x}) = \{\theta : \left(\frac{\sum x_i}{\theta}\right)^m e^{-\frac{\sum x_i}{\theta}} \geq k\}$$

$\Rightarrow$ that the confidence set can be expressed as

$$C(\sum x_i) = \{\theta : L(\sum x_i) \leq \theta \leq U(\sum x_i)\}$$



where

$$\underbrace{\left(\frac{\sum x_i}{L(\sum x_i)}\right)}_{a}^m e^{-\frac{\sum x_i}{L(\sum x_i)}} = \underbrace{\left(\frac{\sum x_i}{U(\sum x_i)}\right)}_{b}^m e^{-\frac{\sum x_i}{U(\sum x_i)}}$$

or

$$a^m e^{-a} = b^m e^{-b}.$$

Further

$$\sum_{i=1}^{m} X_i \sim \Gamma(m, \theta) \implies \frac{\sum X_i}{\theta} \sim \Gamma(m, 1)$$

Also $\dfrac{\sum X_i}{L(\sum X_i)} = a \implies L(\sum X_i) = \dfrac{\sum X_i}{a}$ and similarly $U(\sum x_i) = \dfrac{\sum x_i}{b}$

Hence $P\left(\dfrac{\sum X_i}{a} \le \theta \le \dfrac{\sum X_i}{b} \right) = 1 - \alpha$

$$\iff P\left(b \le \frac{\sum X_i}{\theta} \le a \right) = 1 - \alpha = \int_b^a \frac{1}{\Gamma(m)} t^{m-1} e^{-t} \, dt$$

For $m = 2$ $\quad \displaystyle\int_b^a t e^{-t} dt = e^{-b}(b+1) - e^{-a}(a+1)$

The set of equations
$$a^2 e^{-a} = b^2 e^{-b}$$
$$e^{-b}(b+1) - e^{-a}(a+1) = 1 - \alpha.$$

Can be solved numerically to give $a = 5.48$ and $b = 0.441$ for $\alpha = 0.1$.

which gives the following stochastic interval

$$\left(\frac{\sum_{i=1}^{m} X_i}{5.48} \le \theta \le \frac{\sum_{i=1}^{m} X_i}{0.441} \right)$$

Note $\dfrac{2 \sum_{i=1}^{m} X_i}{\theta} \sim \chi^2(2m)$ and hence

$P\left(\chi^2_{2m, 1-\frac{\alpha}{2}} \le \dfrac{2 \sum X_i}{\theta} \le \chi^2_{2m, \frac{\alpha}{2}} \right) = 1 - \alpha \implies$ a $1-\alpha$ confidence interval is given by

$$\left(\frac{2\sum_{i=1}^{m} X_i}{\chi^2_{2m,\frac{\alpha}{2}}} \leq \theta \leq \frac{2\sum_{1}^{m} X_i}{\chi^2_{2m,1-\frac{\alpha}{2}}} \right)$$

$m = 2$, $\alpha = 0.1$ gives $\left(\frac{\sum_{i=1}^{m} X_i}{4.75}, \frac{\sum_{i=1}^{m} X_i}{0.355} \right)$

Example One sided interval

$X_1, \ldots, X_m$ iid $N(\mu, \sigma^2)$, σ^2 unknown

$$H_0: \mu \geq \mu_0 \quad \text{vs} \quad H_1: \mu < \mu_0$$

Consistency region: $A(\mu_0) = \left\{ \underline{x} : \bar{x} \geq \mu_0 - t_{\alpha, m-1} \frac{s}{\sqrt{m}} \right\}$

$$\Rightarrow C(\underline{x}) = \left\{ \mu : \mu \leq \bar{x} + t_{\alpha, m-1} \frac{s}{\sqrt{m}} \right\}$$

9.2.2. Pivotal Quantities.

Definition 9.2.6. A random variable $Q(\underline{X}, \theta) = Q(X_1, \ldots, X_m, \theta)$ is a pivotal quantity (or pivot) if the distribution of $Q(\underline{X}, \theta)$ is independent of all parameters.

i.e. for any set A: $P(Q(\underline{X}, \theta) \in A)$ is independent of θ.

Examples. $X_1, \ldots, X_m$ iid $N(\mu, \sigma^2)$

$$\left. \begin{array}{l} \dfrac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{m}}} \sim N(0,1) \\ \\ \sigma^2 \text{ unknown} \quad \dfrac{\bar{X} - \mu}{\frac{s}{\sqrt{m}}} \sim t_{m-1} \end{array} \right\} \text{pivots for } \mu$$

$\frac{(m-1)s^2}{\sigma^2} \sim \chi^2(m-1)$ is a pivot for σ^2

$X_1, \ldots, X_n$ iid $\exp(\frac{1}{\theta}) \Rightarrow \dfrac{2\sum X_i}{\theta} \sim \chi^2(n)$ is a pivot

Scale families.

With a standard pdf $f(z)$, the family of pdfs $\frac{1}{\sigma} f(\frac{x}{\sigma})$ is a scale family, and $\dfrac{\bar{X}}{\sigma}$ is a pivot. Proof. $X_i = \sigma Z_i$, $i = 1, 2, \ldots, n \Rightarrow \bar{X} = \sigma \bar{Z} \Rightarrow \dfrac{\bar{X}}{\sigma} = \bar{Z}$ independent of σ.

How to discover a pivot?

Let $f(t) = \dfrac{1}{\Gamma(n)\lambda^n} t^{n-1} e^{-\frac{t}{\lambda}}$, $t > 0$

$$= \frac{1}{\Gamma(n)} \cdot \left(\frac{t}{\lambda}\right)^{n-1} \cdot \frac{1}{\lambda} e^{-\frac{t}{\lambda}} = g\left(\frac{t}{\lambda}\right) \frac{d}{dt}\left(\frac{t}{\lambda}\right)$$

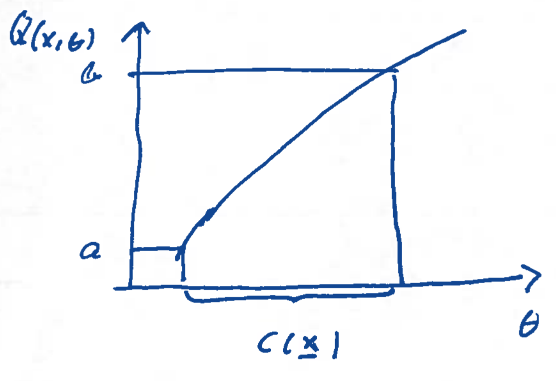
Let $q = \dfrac{t}{\lambda} \Rightarrow t = \lambda q \Rightarrow f_Q(q) = g(q) \cdot \lambda \cdot \dfrac{1}{\lambda} = g(q)$

$\Rightarrow Q = \dfrac{T}{\lambda}$ is a pivot

In general if $f_T(t) = g(h(t)) \dfrac{d}{dt}(h(t))$

and $t = h^{-1}(q)$ is monotone ~~increasing~~ (or $q = h(t)$ is monotone) ~~increasing~~, then $f_Q(q) = g(h(h^{-1}(q))) \cdot \dfrac{dq}{dt} \cdot \dfrac{dt}{dq} = g(q)$

9.2.3. Pivoting the CDF



Confidence set $\{\theta : a \leq Q(\underline{x}, \theta) \leq b\}$ for given $\underline{x}$ is an interval if $Q(\underline{x}, \theta)$ is monotone.

Let T be a statistic, normally sufficient. $Y = F_T(T|\theta)$ is uniform on $(0,1)$, and thereby a pivot.

Thereby $P(\alpha_1 \le F_T(T|\theta) \le 1-\alpha_2) = 1-\alpha_2-\alpha_1$

If $\alpha = \alpha_1 + \alpha_2$, an α-level consistency region of the hypothesis $H_0: \theta = \theta_0$ vs $H_1: \theta \neq \theta_0$ is given by

$A(\theta_0) = \{ t : \alpha_1 \le F_T(t|\theta_0) \le 1-\alpha_2 \}$ with associated confidence set $\{ \theta : \alpha_1 \le F(t|\theta) \le 1-\alpha_2 \}$. given t

Theorem 9.2.12

Let T be a statistic with continuous cdf. $\alpha_1 + \alpha_2 = \alpha$

Suppose for each $t \in \mathcal{T}$, $\theta_L(t)$ and $\theta_U(t)$ can be defined as follows

1. If $F_T(t|\theta)$ is a decreasing function of θ for each θ, define $\theta_L(t)$ and $\theta_U(t)$ by Eks. normalfordeling.

$$F_T(t|\theta_U(t)) = \alpha_1, \quad F_T(t|\theta_L(t)) = 1-\alpha_2$$

2. If $F_T(t|\theta)$ is an increasing function of θ for each t, define $\theta_L(t)$ and $\theta_U(t)$ by Eks. exp $f(x|\mu) = e^{-(x-\mu)} I_{[\mu,\infty)}$

$$F_T(t|\theta_U(t)) = 1-\alpha_2, \quad F_T(t|\theta_L(t)) = \alpha_1$$

Then $[\theta_L(t), \theta_U(t)]$ is a $1-\alpha$ confidence interval for θ.