$$f_{U,V}(u,v) = f_{X,Y}(h_1(u,v), h_2(u,v)) \, |J|$$

$$= \frac{1}{2\pi} e^{-\frac{\left(\frac{u+v}{2}\right)^2}{2}} \cdot e^{-\frac{\left(\frac{u-v}{2}\right)^2}{2}} \cdot \left|-\tfrac{1}{2}\right|$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sqrt{2}} e^{-\frac{u^2}{4}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2}} e^{-\frac{v^2}{4}} = g(u)\cdot g(v)$$

$\Rightarrow$ U and V are independent $\sim N(0, \sqrt{2}^2)$

<u>Theorem 4.3.5</u>

X and Y independent $\Rightarrow$ $U = g(X)$ and $V = h(Y)$ are independent.

<u>Proof.</u> For any $u \in R$ and $v \in R$, let

$A_u = \{x : g(x) \leq u\}$ and $B_v = \{y : h(y) \leq v\}$

$$F_{U,V}(u,v) = P(U \leq u \cap V \leq v) = P(X \in A_u \cap Y \in B_v)$$

$= P(X \in A_u) \cdot P(Y \in B_v)$ and $f_{U,V}(u,v) = \frac{\partial^2}{\partial u \partial v} F_{U,V}(u,v)$

$$= \frac{\partial}{\partial u} P(X \in A_u) \cdot \frac{\partial}{\partial v} P(Y \in B_u)$$

<u>Example 4.3.6</u>

X and Y independent $\sim N(0,1)$

$U = \frac{X}{Y}$, $V = |Y|$

Then (X,Y) and $(-X,-Y)$ gives the same U and V and

The transformation is not one to one.

$y < 0 \Rightarrow y = -v$ and $x = -uv \Rightarrow J = \begin{vmatrix} -v & -u \\ 0 & -1 \end{vmatrix} = v$

$y > 0 \Rightarrow y = v$, $x = uv$ and $J = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{x^2}{2}} \cdot e^{-\frac{y^2}{2}}$$

Such that

$$F_{U,V}(u,v) = \overbrace{\int_0^v \int_{-\infty}^u \frac{1}{2\pi} e^{-\frac{(-st)^2}{2}} \cdot e^{-\frac{(-t)^2}{2}} \cdot t \, ds\, dt}^{y<0} + \overbrace{\int_0^v \int_{-\infty}^u \frac{1}{2\pi} e^{-\frac{(st)^2}{2}} e^{-\frac{t^2}{2}} t \, ds\, dt}^{y>0}$$

$$= \int_0^v \int_{-\infty}^u \frac{t}{\pi} e^{-(s^2+1)\frac{t^2}{2}} ds\, dt$$

$$\Rightarrow f_{U,V}(u,v) = \frac{v}{\pi} e^{-(u^2+1)\cdot\frac{v^2}{2}}, \quad -\infty < u < \infty, \quad v > 0$$

$$f_U(u) = \int_0^\infty \frac{v}{\pi} e^{-(u^2+1)\frac{v^2}{2}} dv = \left[\frac{-e^{-(u^2+1)\frac{v^2}{2}}}{\pi(u^2+1)} \right]_0^\infty = \frac{1}{\pi(u^2+1)}$$

$-\infty < u < \infty$

i.e. Cauchy distributed.

4.4 Hierarchical models and Mixture distributions

Example.

Y is the number of eggs laid of an insect mother

$X \sim$ is the number of survivors.

Reasonable $Y \sim Po(\lambda)$ and $X/Y \sim Binomial(Y, p)$

Example 4.4.2

$$P(X=x) = \sum_{y=0}^{\infty} P(X=x \cap Y=y) = \sum_{y=0}^{\infty} P(X=x \mid Y=y) \cdot P(Y=y)$$

$$= \sum_{y=x}^{\infty} \left[\binom{y}{x} p^x (1-p)^{y-x} \cdot \frac{e^{-\lambda} \lambda^y}{y!} \right]$$

$$= \sum_{y=x}^{\infty} \frac{y!}{x!(y-x)!} p^x (1-p)^{y-x} \frac{e^{-\lambda} \lambda^y}{y!}$$

$$= \frac{e^{-\lambda} (\lambda p)^x}{x!} \sum_{y=x}^{\infty} \frac{((1-p)\lambda)^{y-x}}{(y-x)!} \overset{k=y-x}{=} \frac{e^{-\lambda} (\lambda p)^x}{x!} \sum_{k=0}^{\infty} \frac{((1-p)\lambda)^k}{k!}$$

$$= \frac{e^{-\lambda} (\lambda p)^x}{x!} e^{(1-p)\lambda} = \frac{(\lambda p)^x e^{-\lambda p}}{x!} \sim Po(\lambda p)$$

Definition 4.4

A random variable X, is said to have a mixture distribution if the distribution of X depends on a quantity that also has a distribution.

Example 4.4.5

A large number of mothers with varying λ's

Y is the number of eggs laid by one mother chosen at random. X is the number of survivors.

$$X|Y \sim \text{Binomial}(Y, p)$$

$$Y|\Lambda \sim \text{Poisson}(\Lambda) \qquad X|\Lambda \sim Po(p\Lambda)$$

$$\Lambda \sim \text{exponential}\left(\frac{1}{\beta}\right)$$

$$P(Y=y) = \int_0^\infty f(y,\lambda)\, d\lambda = \int_0^\infty f(y|\lambda) f(\lambda)\, d\lambda$$

$$= \int_0^\infty \frac{e^{-\lambda}\lambda^y}{y!} \cdot \frac{1}{\beta} e^{-\frac{\lambda}{\beta}}\, d\lambda = \frac{1}{\beta y!}\int_0^\infty \lambda^y e^{-\lambda(1+\frac{1}{\beta})}\, d\lambda$$

$$\overset{t=\lambda(1+\frac{1}{\beta})}{=} \frac{1}{\beta y!}\int_0^\infty \frac{t^y}{(1+\frac{1}{\beta})^{y+1}} e^{-t}\, dt = \frac{1}{\beta y!} \cdot \frac{1}{(1+\frac{1}{\beta})^{y+1}} \Gamma(y+1)$$

$$= \frac{1}{1+\beta} \cdot \frac{1}{(1+\frac{1}{\beta})^y} = \frac{1}{1+\beta}\left(1 - \frac{1}{1+\beta}\right)^y$$

Negative binomial

$$P(X=x \mid r, p) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \quad x = r, r+1, \ldots$$

$$\overset{Y=X-r}{=} \binom{r+y-1}{r-1} p^r (1-p)^y, \quad y = 0, 1, 2, \ldots$$

with $r=1$ and $p = \frac{1}{1+\beta}$, Y follows a translated negative binomial.

Conditional expectation and variance

Theorem 4.4.3

X, Y random variables. Then $E[X] = E[E[X|Y]]$ provided the expectation exist

Proof.

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} x f(x|y)\, dx \right)}_{E[X|Y]} f_Y(y)\, dy = E[E(X|Y)]$$

Theorem 4.4.7

X, Y random variables. Then

$Var[X] = E[Var(X|Y)] + Var[E(X|Y)]$ provided it exist

Proof.

$$Var[X] = E[X^2] - (E[X])^2.$$

$$= E[E[X^2|Y]] - (E[E(X|Y)])^2.$$

$$= E[Var[X|Y] + (E[X|Y])^2] - (E[E(X|Y)])^2.$$

$$= E[Var[X|Y]] + Var[E[X|Y]]$$

Some examples. One mother.

$Y \sim Po(\lambda)$, $\quad X|Y \sim B(Y, p)$

$E[X|Y] = pY \Rightarrow \quad E[X] = E[E[X|Y]] = pE[Y] = p\lambda$

$Var[X|Y] = Yp(1-p) \Rightarrow Var[X] = \lambda p(1-p) + p^2\lambda$

$= \lambda p(1-p+p) = \lambda p$

<u>Randomly chosen mother</u>

$X|Y \sim B(Y, p)$

$Y/\Lambda \sim Po(\Lambda)$

$\Lambda \sim exp(\frac{1}{\beta})$

$E[X] = E[E[X|Y]] = E[pY] = pE[Y] = pE[E(Y|\Lambda)] = pE[\Lambda] = p\beta$

$Var[X] = E[Var(X|Y)] + Var[E[X|Y]]$

$= E[Yp(1-p)] + Var[pY]$

$= p(1-p)E[Y] + p^2 Var[Y]$

$= p(1-p)E[E[Y|\Lambda]] + p^2[E[Var(Y|\Lambda)] + Var[E(Y|\Lambda)]]$

$= p(1-p)E[\Lambda] + p^2[E[\Lambda] + Var[\Lambda]]$

$= p\beta - p^2\beta + p^2(\beta + \beta^2) = p\beta(1+p\beta)$

For covariance

$E[XY] = E[E[XY|X]] = E[X E[Y|X]]$.