Thereby $\quad P(\alpha_1 \le F_T(T|\theta) \le 1-\alpha_2) = 1-\alpha_2-\alpha_1$

If $\alpha = \alpha_1+\alpha_2$, is an α-level consistency region of the hypothesis $H_0: \theta = \theta_0$ vs $H_1: \theta \neq \theta_0$ is given by

$A(\theta_0) = \{ t : \alpha_1 \le F_T(t|\theta_0) \le 1-\alpha_2 \}$ with associated confidence set $\{ \theta : \alpha_1 \le F(t|\theta) \le 1-\alpha_2 \}$. given t

<u>Theorem 9.2.12</u>

Let T be a statistic with continuous cdf. $\alpha_1+\alpha_2 = \alpha$

Suppose for each $t \in \mathcal{T}$, $\theta_L(t)$ and $\theta_U(t)$ can be defined as follows

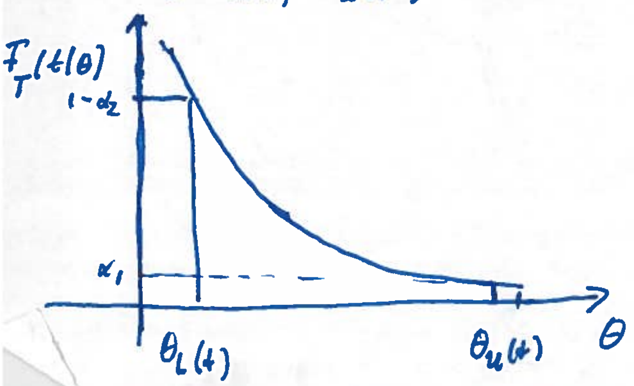
1. If $F_T(t|\theta)$ is a decreasing function of θ for each θ, define $\theta_L(t)$ and $\theta_U(t)$ by $\qquad$ Uns. normalfordeling.

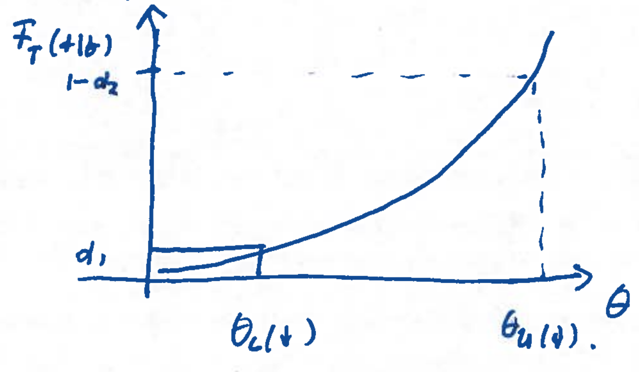
$$F_T(t|\theta_U(t)) = \alpha_1, \qquad F_T(t|\theta_L(t)) = 1-\alpha_2$$

2. If $F_T(t|\theta)$ is an increasing function of θ for each t, define $\theta_L(t)$ and $\theta_U(t)$ by $\qquad$ Uns. exp $\cancel{f(x|\theta) = e^{(\ldots)}}\, I_{[0,\infty)}$ eksponential fordeling.

$$F_T(t|\theta_U(t)) = 1-\alpha_2, \qquad F_T(t|\theta_L(t)) = \alpha_1 \quad \left| \begin{array}{ll} F(x) = 1-e^{-\lambda x} & \text{increasing} \\ F(x) = 1-e^{-\frac{1}{\beta}x} & \text{decreasing.} \end{array} \right.$$

Then $[\theta_L(T), \theta_U(T))$ is a $1-\alpha$ confidence interval for θ.





In case 1 we have

$$P(T \leq t \mid \theta_U(t)) = \int_{-\infty}^{t} f_T(s \mid \theta_U(t))\, ds = \alpha_1$$

$$P(T \geq t \mid \theta_L(t)) = \int_{t}^{\infty} f_T(s \mid \theta_L(t))\, ds = \alpha_2$$

stochastic increasing since $\{X_1, \ldots, X_m \mid \theta_2\} \geq \{X_1, \ldots, X_m \mid \theta_1\}$ if $\theta_2 > \theta_1$

In case 2 we have

$$P(T \leq t \mid \theta_L(t)) = \int_{-\infty}^{t} f_T(s \mid \theta_L(t))\, ds = \alpha_1$$

$$P(T \geq t \mid \theta_U(t)) = \int_{t}^{\infty} f_T(s \mid \theta_U(t))\, ds = \alpha_2$$

stochastic decreasing.

Read 9.2.13

Theorem 9.2.14 Pivoting a discrete cdf.

Let T be a discrete statistic with cdf $F_T(t|\theta) = P(T \leq t|\theta)$

Let $\alpha_1 + \alpha_2 = \alpha$ with $0 < \alpha < 1$. Suppose $\forall t \in \mathcal{T}$, $\theta_L(t)$ and $\theta_U(t)$ can be defined as

1. If $F_T(t|\theta)$ is a decreasing function of θ for each t define $\theta_L(t)$ and $\theta_U(t)$ as   Example Poisson, Binomial

$$P(T \leq t \mid \theta_U(t)) = \alpha_1 \quad , \quad P(T \geq t \mid \theta_L(t)) = \alpha_2 .$$

2. $F_T(t|\theta)$ is an increasing function of θ for each t, define $\theta_L(t)$ and $\theta_U(t)$ by $P(T \geq t \mid \theta_U(t)) = \alpha_2$

$P(T \leq t \mid \theta_L(t)) = \alpha_1$ Example. Geometrical distribution $P(X \leq k) = 1-(1-p)^k$

Then the random interval $[\theta_L(T), \theta_U(T)]$ is a $1-\alpha$ confidence interval for θ.

Example 9.2.15

$X_1, \dots, X_n$ iid Poisson(λ) $Y = \sum_{i=1}^{n} X_i \sim$ Poisson$(n\lambda)$

Assume $\alpha_1 = \alpha_2 = \frac{\alpha}{2}$

$F_Y(t) = P(Y \leq t)$ is a decreasing function of λ

If $Z \sim$ gamma(α, β), α integer, Z models the time to event α in the poisson process and

$$P(Z \leq z) = P(Y \geq \alpha), \text{ where } Y \sim \text{Poisson}\left(\frac{z}{\beta}\right)$$

expected time between events

Expected number pr. time unit $\frac{1}{\beta}$ and $\frac{z}{\beta} = m\lambda$

For λ_u we get, when the observed value for $Y = y_0$

$$\frac{\alpha}{2} = \sum_{k=0}^{y_0} e^{-m\lambda}\frac{(m\lambda)^k}{k!} = 1 - P(Y \geq y_0+1) = 1 - P(Z \leq z) = P(Z > z)$$

$$= P\left(\frac{2Z}{\beta} \geq \frac{2z}{\beta} = 2m\lambda\right)$$

$$\frac{2Z}{\beta} \sim \chi^2(2\alpha) \sim \chi^2(2(y_0+1))$$

Hence $P(Z > z) = P(\chi^2(2(y_0+1)) > 2m\lambda_u)$

Thereby $2m\lambda_u = \chi^2(2(y_0+1))_{\frac{\alpha}{2}}$ and $\lambda_u = \frac{1}{2m}\chi^2(2(y_0+1)),_{\frac{\alpha}{2}}$

<u>For λ_L</u>

$$\frac{\alpha}{2} = \sum_{k=y_0}^{\infty} e^{-m\lambda}\frac{(m\lambda)^k}{k} = P(Y \geq y_0) = P(Z \leq z)$$

$$= P\left(\frac{2Z}{\beta} \leq \frac{2z}{\beta}\right) = P(\chi^2(2y_0) \leq 2m\lambda)$$

$$\Rightarrow 2m\lambda_L = \chi^2(2y_0)_{1-\frac{\alpha}{2}} \text{ and } \lambda_L = \frac{1}{2m}\chi^2(2(y_0))_{1-\frac{\alpha}{2}}$$

We get the interval

$$\left\{\lambda : \frac{1}{2m}\chi^2(2y_0),_{1-\frac{\alpha}{2}} \leq \lambda \leq \frac{1}{2m}\chi^2(2(y_0+1)),_{\frac{\alpha}{2}}\right\}$$

9.2.4 Bayesian Intervals

Prior $\pi(\theta)$, likelihood (joint density) $f(\underline{x}|\theta)$

Posterior $\pi(\theta|\underline{x}) = \dfrac{f(\underline{x},\theta)}{m(\underline{x})} = \dfrac{f(\underline{x}|\theta)\,\pi(\theta)}{\int f(\underline{x},\theta)\,d\theta}$

θ is a random variable

Let $A \subset \Omega$ (the parameter space for θ)

Can compute $P(\theta \in A|\underline{x}) = \int_A \pi(\theta|\underline{x})\,d\theta$

as the credible probability of A and A is called the credible set for θ.

Example.

$X_1, \dots, X_n$ i.i.d $Po(\lambda) \Rightarrow Y = \sum_{i=1}^{n} X_i \sim Po(n\lambda)$

$\pi(\lambda) = \dfrac{1}{\Gamma(\alpha)\,\beta^\alpha}\lambda^{\alpha-1} e^{-\frac{\lambda}{\beta}}$, i.e $\Gamma(\alpha,\beta)$

$f(y,\lambda) = \dfrac{(n\lambda)^y e^{-n\lambda}}{y!} \cdot \dfrac{1}{\Gamma(\alpha)}\dfrac{\lambda^{\alpha-1}}{\beta^\alpha} e^{-\frac{\lambda}{\beta}}$

$$= \underbrace{\frac{\lambda^{y+\alpha-1}\, e^{-\lambda(n+\frac{1}{\beta})}}{\Gamma(\alpha+\beta)\cdot\left(\frac{\beta}{\beta n+1}\right)^{y+\alpha}}}_{\pi(\lambda|y) \sim \Gamma\left(y+\alpha,\, \frac{\beta}{\beta n+1}\right)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \cdot \frac{n^y}{y!} \cdot \frac{\beta^y}{(\beta n+1)^{y+\alpha}}.$$