9.2.4 Bayesian Intervals

Prior $\pi(\theta)$, likelihood (joint density) $f(\underline{x}|\theta)$

Posterior $\pi(\theta|\underline{x}) = \dfrac{f(\underline{x},\theta)}{m(\underline{x})} = \dfrac{f(\underline{x}|\theta)\,\pi(\theta)}{\int f(\underline{x},\theta)\,d\theta}$

θ is a random variable

Let $A \subset \Omega$ (the parameter space for θ)

Can compute $P(\theta \in A|\underline{x}) = \int_A \pi(\theta|\underline{x})\,d\theta$

as the credible probability of A and A is called the credible set for θ.

Example.

$X_1, \dots, X_n$ i.i.d $Po(\lambda) \Rightarrow Y = \sum_{i=1}^{n} X_i \sim Po(n\lambda)$

$\pi(\lambda) = \dfrac{1}{\Gamma(\alpha)\,\beta^\alpha}\,\lambda^{\alpha-1} e^{-\frac{\lambda}{\beta}}$, i.e $\Gamma(\alpha,\beta)$

$f(y,\lambda) = \dfrac{(n\lambda)^y e^{-n\lambda}}{y!} \cdot \dfrac{1}{\Gamma(\alpha)} \dfrac{\lambda^{\alpha-1}}{\beta^\alpha} e^{-\frac{\lambda}{\beta}}$

$$= \underbrace{\frac{\lambda^{y+\alpha-1}\, e^{-\lambda(n+\frac{1}{\beta})}}{\Gamma(\alpha+y)\cdot\left(\frac{\beta}{\beta n+1}\right)^{y+\alpha}}}_{\pi(\lambda|y) \sim \Gamma\left(y+\alpha,\ \frac{\beta}{\beta n+1}\right)} \cdot \frac{\Gamma(\alpha+y)}{\Gamma(\alpha)} \cdot \frac{n^y}{y!} \cdot \frac{\beta^y}{(\beta n+1)^{y+\alpha}}.$$

Therefore $\dfrac{2(\beta m+1)}{\beta}\lambda | y \sim \chi^2(2(y+\alpha))$

Given $Y=y$ we have

$$P\left(\chi^2(2(y+\alpha))_{1-\frac{\alpha}{2}} \le \frac{2(\beta m+1)}{\beta}\lambda \le \chi^2(2(y+\alpha))_{\frac{\alpha}{2}}\right) = 1-\alpha$$

and a $1-\alpha$ credible set is given by:

$$\left\{\lambda : \frac{\beta}{2(\beta m+1)}\chi^2(2(y+\alpha))_{1-\frac{\alpha}{2}} \le \lambda \le \frac{\beta}{2(\beta m+1)}\chi^2(2(y+\alpha))_{\frac{\alpha}{2}}\right\}$$

<u>Example 9.2.18</u>

$X_1, \dots, X_m$ iid $N(\theta, \sigma^2)$, $\pi(\theta) \sim N(\mu, \tau^2)$, μ, σ^2 and τ^2 are known.

$$\pi(\theta | \underline{x}) \sim N(\delta^B(\bar{x}), Var(\theta|\bar{x}))$$

where $\delta^B(\bar{x}) = \dfrac{\sigma^2}{\sigma^2+m\tau^2}\mu + \dfrac{m\tau^2}{\sigma^2+m\tau^2}\bar{x}$, $Var(\theta|\bar{x}) = \dfrac{\sigma^2\tau^2}{\sigma^2+m\tau^2}$

θ is random and $\bar{x}$ is fixed

Hence $\dfrac{\theta-\delta^B(\bar{x})}{\sqrt{Var(\theta|\bar{x})}} \sim N(0,1)$

and a $1-\alpha$ credibility interval is given by

$$\left(\delta^B(\bar{x}) - z_{\frac{\alpha}{2}}\sqrt{Var(\theta|\bar{x})} \le \theta \le \delta^B(\bar{x}) + z_{\frac{\alpha}{2}}\sqrt{Var(\theta|\bar{x})}\right)$$

The coverage probability is

$$P\left(|\theta - \delta^B(\bar{x})| \le z_{\frac{\alpha}{2}}\sqrt{Var(\theta|\bar{x})}\right)$$

where now θ is fixed and $\bar{X}$ is random and $Var(\bar{X}) = \dfrac{\sigma^2}{m}$

Introduce $\gamma = \frac{\sigma^2}{m\tau^2} \Rightarrow \begin{cases} \delta^B(\bar{x}) = \dfrac{\frac{\sigma^2}{m\tau^2}}{\frac{\sigma^2}{m\tau^2}+1}\mu + \dfrac{\frac{m\tau^2}{m\tau^2}\bar{x}}{\frac{\sigma^2}{m\tau^2}+1} = \dfrac{\gamma}{1+\gamma}\mu + \dfrac{\bar{x}}{1+\gamma} \\ Var(\theta|\bar{x}) = \dfrac{\frac{\sigma^2\tau^2}{m\tau^2}}{\frac{\sigma^2}{m\tau^2}+\frac{m\tau^2}{m\tau^2}} = \dfrac{\sigma^2}{m(1+\gamma)} \end{cases}$

and the coverage probability becomes.

$$P\left(\left|\theta - \left(\frac{\gamma}{1+\gamma}\mu + \frac{1}{1+\gamma}\bar{x}\right)\right| \le z_{\frac{\alpha}{2}}\sqrt{\frac{\sigma^2}{m(1+\gamma)}} \right)$$

$$= P\left(-z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{m}}\cdot\frac{1}{\sqrt{1+\gamma}} \le \theta - \frac{\gamma}{1+\gamma}\mu - \frac{\bar{x}}{1+\gamma} \le z_{\frac{\alpha}{2}}\cdot\frac{\sigma}{\sqrt{m}}\cdot\frac{1}{\sqrt{1+\gamma}}\right)$$

$$= P\left(-z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{m}}\sqrt{1+\gamma} \le \theta(1+\gamma) - \gamma\mu - \bar{x} \le z_{\frac{\alpha}{2}}\cdot\frac{\sigma}{\sqrt{m}}\sqrt{1+\gamma}\right)$$

$$= P\left(-z_{\frac{\alpha}{2}}\sqrt{1+\gamma} + \frac{\gamma(\mu-\theta)}{\frac{\sigma}{\sqrt{m}}} \le \frac{\theta-\bar{x}}{\frac{\sigma}{\sqrt{m}}} \le \frac{\gamma(\mu-\theta)}{\frac{\sigma}{\sqrt{m}}} + z_{\frac{\alpha}{2}}\sqrt{1+\gamma}\right)$$

$$\Rightarrow P\left(-z_{\frac{\alpha}{2}}\sqrt{1+\gamma} + \frac{\gamma(\theta-\mu)}{\frac{\sigma}{\sqrt{m}}} \le Z \le \frac{\gamma(\theta-\mu)}{\frac{\sigma}{\sqrt{m}}} + z_{\frac{\alpha}{2}}\sqrt{1+\gamma}\right)$$

Choose $\gamma = 1 \left(\tau = \frac{\sigma}{\sqrt{m}}\right)$, $\theta > \mu \Rightarrow -z_{\frac{\alpha}{2}}\sqrt{2} + \frac{\theta-\mu}{\frac{\sigma}{\sqrt{m}}} \to \infty$ as $m \to \infty$

$\Rightarrow$ the coverage probability goes to zero. If $\theta = \mu$ it is bounded away from zero.

<u>On the other side</u>

Start with a $1-\alpha$ confidence interval.

$$\bar{x} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{m}} \le \theta \le \bar{x} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{m}}$$

θ is random and $\bar{x}$ fixed $\overset{\theta \sim \pi(\theta|\bar{x})}{\text{gives}}$.

Need to find

$$P\left(|\theta - \delta^B(\bar{x}) + \delta^B(\bar{x}) - \bar{x}| \le z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right)$$

$$\delta^B(\bar{x}) - \bar{x} = \frac{\gamma\mu}{1+\gamma} - \frac{\gamma\bar{x}}{1+\gamma} = \frac{\gamma(\mu - \bar{x})}{1+\gamma}$$

We get

$$P\left(\frac{-z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}}{\frac{\sigma}{\sqrt{n(1+\gamma)}}} + \frac{\frac{\gamma(\bar{x}-\mu)}{\sigma(1+\gamma)}}{\sqrt{n(1+\gamma)}} \le \frac{\theta - \delta^B(\bar{x})}{\frac{\sigma}{\sqrt{n(1+\gamma)}}} \le \frac{\frac{\gamma(\bar{x}-\mu)}{\sigma(1+\gamma)}}{\sqrt{n(1+\gamma)}} + \frac{z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}}{\frac{\sigma}{\sqrt{n(1+\gamma)}}} \right)$$

$$= P\left(-\sqrt{1+\gamma}\, z_{\frac{\alpha}{2}} + \frac{\gamma(\bar{x}-\mu)}{\sqrt{1+\gamma}\, \frac{\sigma}{\sqrt{n}}} \le Z \le \frac{\gamma(\bar{x}-\mu)}{\sqrt{1+\gamma} \cdot \frac{\sigma}{\sqrt{n}}} + z_{\frac{\alpha}{2}} \sqrt{1+\gamma} \right)$$

If $\gamma = 1$ ($\tau = \frac{\sigma}{\sqrt{n}}$) and $\bar{x} > \mu$, the credible probability $\to 0$ as $\frac{\sigma}{\sqrt{n}} \to 0$

Credible probability reflects the experimenter's subjective belief updated with the data

Coverage — " — reflects the uncertainty in the sampling procedure

Chapter 10. Asymptotic Evaluation

Definition 10.1.1

A sequence of estimators $W_n = W_n(X_1, \dots, X_n)$ is a consistent sequence of estimators of θ if $\forall \varepsilon > 0$ and every $\theta \in \Omega$,

$$\lim_{n \to \infty} P(|W_n - \theta| < \varepsilon) = 1 \Leftrightarrow \lim_{n \to \infty} P(|W_n - \theta| \ge \varepsilon) = 0$$

We have from Chebychev

$$P(|W_n - \theta| \ge \varepsilon) \le \frac{E[(W_n - \theta)^2]}{\varepsilon^2} = \frac{\mathrm{Var}[W_n]}{\varepsilon^2} + \frac{(\mathrm{Bias}\, W_n)^2}{\varepsilon^2}.$$

Theorem 10.1.3

A sequence of estimators for θ satisfying

1. $\lim_{n \to \infty} \mathrm{Var}[W_n] = 0$
2. $\lim_{n \to \infty} \mathrm{Bias}[W_n] = 0$

$\forall \theta \in \Omega$ is a consistent sequence of estimators.

Theorem 10.1.6

$X_1, \dots, X_n$ iid. Let $\hat{\theta}$ be the MLE of θ. Let $\tau(\theta)$ be a continuous function of θ. Under certain regularity conditions on $f(x|\theta)$ (10.6.2, A1–A4 p. 516) we have $\forall \varepsilon > 0$ and every $\theta \in \Omega$ that $\lim\limits_{n\to\infty} P(|\tau(\hat{\theta}) - \tau(\theta)| \geq \varepsilon) = 0$

i.e $\tau(\hat{\theta})$ is a consistent estimator for $\tau(\theta)$

Definition 10.1.7

For an estimator T_n, if $\lim\limits_{n\to\infty} k_n \operatorname{Var}(T_n) = \tau^2 < \infty$, τ^2 is called a limiting variance.

If $k_n(T_n - \tau(\theta)) \xrightarrow{D} N(0, \sigma^2)$, σ^2 is called the asymptotic variance.

Example. Random sample $X_1, \dots, X_n$, $E[X_i] = \mu$, $0 < \operatorname{Var} X_i < \infty$, $i = 1, 2, \dots, n$

Then MLE for $\mu = \bar{X}_n$ and from CLT $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{D} N(0, \sigma^2)$

Also $\frac{1}{\bar{X}_n}$ is the MLE of $\frac{1}{\mu}$ and from the Delta theorem

$$\sqrt{n}\left(\frac{1}{\bar{X}_n} - \frac{1}{\mu}\right) \xrightarrow{D} N\left(0, \frac{1}{\mu^4}\sigma^2\right)$$ which shows that the

asymptotic variance of $T_n = \frac{1}{\bar{X}_n}$ exist, $\mu \neq 0$

However. The exact variance of $\frac{1}{\bar{X}_n} = \infty$, $\forall n$ and the

limiting variance does not exist.