Enough with two walls. The two at the end can be removed

M I I M   I M M I   M M I I

1 3   2 2   1 1

Two walls $(m-1)$ and two numbers can be arranged in $(3-1+2)!$ $\quad ((m-1+r)!)$ ways.

To remove double counting we get.

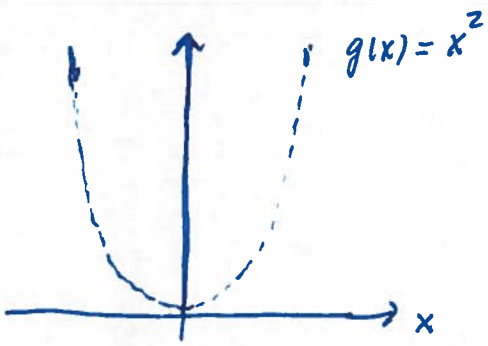
$$\frac{4!}{2!\,2!} \quad \text{or} \quad \frac{(m-1+r)!}{(m-1)!\,r!} = \binom{m-1+r}{r}$$

Chapter 2. Transformations and Expectation

$$Y = g(X) : \qquad g(x) : \mathcal{X} \to \mathcal{Y} .$$

$$F_Y(y) = P(g(X) \le y) = P(x \in \mathcal{X} : g(x) \le y) \overset{\text{cont.}}{=} \int_{\{x :\, g(x) \le y\}} f_X(x)\,dx$$

Example 2.1.7



X continuous with support set $\mathcal{X} = (-\infty, \infty)$

$$Y = g(X) = X^2$$

$$F_Y(y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= P(X \le \sqrt{y}) - P(X \le -\sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$\Rightarrow f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})$$

In particular let $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $x \in (-\infty, \infty)$

o; $X \sim N(0,1)$

$$Y = X^2 \Rightarrow f_Y(y) = \frac{1}{2\sqrt{y}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{y})^2}{2}} + \frac{1}{2\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(-\sqrt{y})^2}{2}}$$

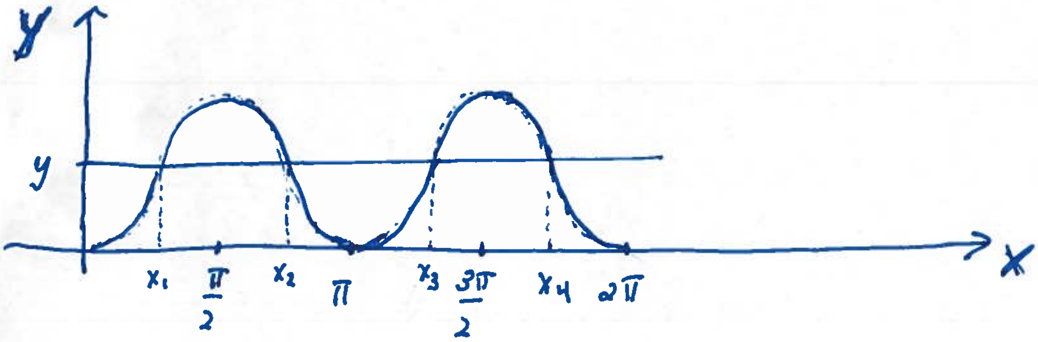
$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{y}} e^{-\frac{y}{2}}, \quad 0 < y < \infty \qquad \text{o; } Y \sim \chi^2(1)$$

Example 2.1.2

X cont, uniform on $(0, 2\pi)$

i.e. $f_X(x) = \begin{cases} \frac{1}{2\pi} & 0 < x < 2\pi \\ 0 & \text{otherwise} \end{cases}$, $\quad F_X(x) = \frac{x}{2\pi}$, $0 < x < 2\pi$

$Y = \sin^2 X$

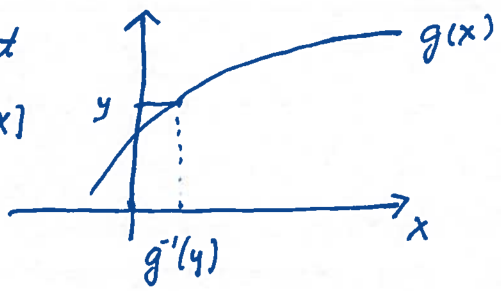


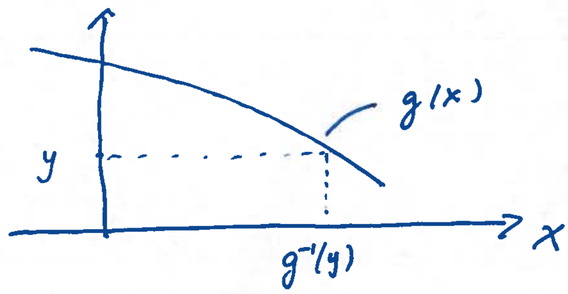
$$P(Y \le y) = P(X \le x_1) + P(x_2 \le X \le x_3) + P(X \ge x_4) = F_X(x_1)$$
$$+ (F_X(x_3) - F_X(x_2)) + 1 - F_X(x_4) = \frac{1}{2\pi}\{x_1 + x_3 - x_2 - x_4 + 2\pi\}$$

Monotone transformation. Theorem 2.1.5

X cont

$Y = g(X)$





$F_Y(y) = P(Y \le y) = P(g(X) \le y)$
$= P(X \le g^{-1}(y)) = F_X(g^{-1}(y))$

$F_Y(y) = P(Y \le y) = P(g(X) \le y)$
$= 1 - P(X \ge g^{-1}(y)) = 1 - F_X(g^{-1}(y))$

We get $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$

A special transformation 2.1.10

X cont. $F_X(x) = P(X \le x)$. $F_X(x)$ is monotone

Let $Y = F_X(x) = P(Y \le y) = P(F_X(x) \le y) = P(X \le F_X^{-1}(y))$

$= F_X(F_X^{-1}(y)) = y$ i.e Y is uniform on $[0,1]$.

Let Y be uniform on $[0,1]$. Let $X = F_X^{-1}(Y)$

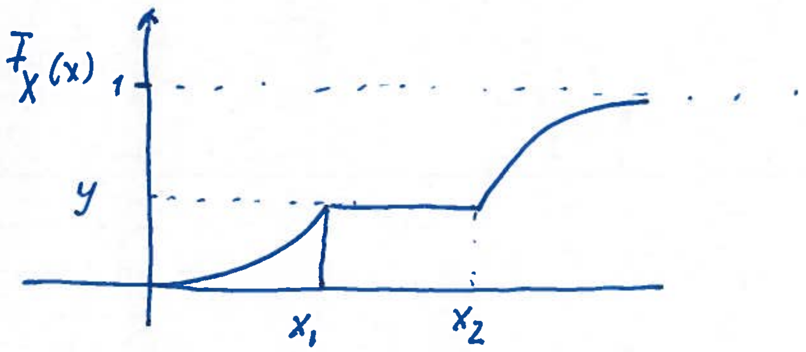
$\Rightarrow P(X \le x) = P(F_X^{-1}(Y) \le x) = P(Y \le F_X(x)) = F_X(x)$

Example $X \sim \exp(\lambda) \Rightarrow F_X(x) = 1 - e^{-\lambda x}$, $x > 0$

Y cont. unif $[0,1]$. $y = F_X(x) \Rightarrow 1 - e^{-\lambda x} = y$

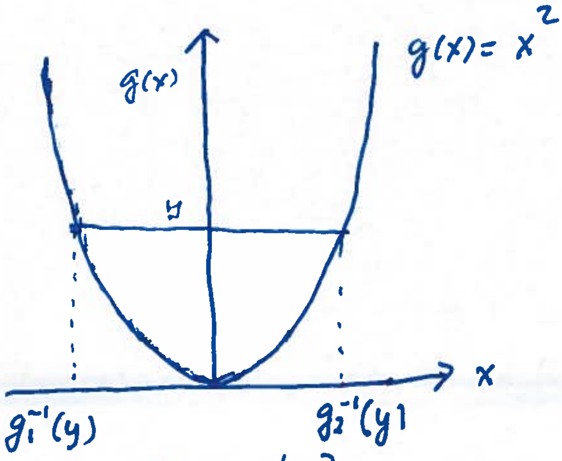
$\Rightarrow e^{-\lambda x} = 1 - y \Rightarrow x = -\frac{1}{\lambda} \ln(1-y)$

What if $F_X(x)$ is constant on some interval?



$F_X^{-1}(y) = \inf\{x : F_X(x) \ge y\}$ ɔ: x_1

Example 2.1.7 reconsidered



$$Y = g(X) = X^2$$

Define: $A_0 = \{0\}$

$A_1 = (-\infty, 0), \quad g_1(x) = x^2, \quad g_1^{-1}(y) = -\sqrt{y}$

$A_2 = (0, \infty), \quad g_2(x) = x^2, \quad g_2^{-1}(y) = \sqrt{y}$

$\mathcal{Y} = \{y : y = g_i(x)\} = (0, \infty)$ for $i = 1, 2$

$$F_Y(y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y})$$

$$= P(X \leq \sqrt{y}) - P(X \leq -\sqrt{y}) = F_X(g_2^{-1}(y)) - F_X(g_1^{-1}(y))$$

$$\Rightarrow f_Y(y) = f_X(g_2^{-1}(y)) \frac{d}{dy} g_2^{-1}(y) - f_X(g_1^{-1}(y)) \frac{d}{dy} g_1^{-1}(y)$$

$$= \sum_{i=1}^{2} f_X(g_i^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right|$$

2.2. Expectation

$$E[X] = \begin{cases} \int_{-\infty}^{\infty} x f_X(x)\, dx \\ \sum x \, P(X=x) \end{cases} \quad \text{if} \quad \begin{cases} \int_{-\infty}^{\infty} |x| f_X(x)\, dx < \infty \\ \sum |x| \, P(X=x) < \infty \end{cases}$$

Example 2.24

$$f_X(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$$

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)} dx = \int_{-\infty}^{0} \frac{x}{\pi(1+x^2)} dx + \int_{0}^{\infty} \frac{x}{\pi(1+x^2)} dx$$

$$= \int_{\infty}^{0} \frac{-y}{\pi(1+y^2)} -dy + \int_{0}^{\infty} \frac{x}{\pi(1+x^2)} dx$$

$$= -\int_{0}^{\infty} \frac{y}{\pi(1+y^2)} dy + \left[\frac{\log(1+x^2)}{2\pi} \right]_0^{\infty}$$

$$= -\infty + \infty$$

$$E[|X|] = \int_{-\infty}^{\infty} \frac{|x|}{\pi} \cdot \frac{1}{1+x^2} dx = 2 \int_{0}^{\infty} \frac{x}{\pi} \frac{1}{1+x^2} dx = \infty \implies$$

$E[X]$ does not exist.

Important results

$$E[g(X)] = \begin{cases} \int_{-\infty}^{\infty} g(x) f_X(x)\, dx \\ \sum_{x \in \mathcal{X}} g(x) \, P(X=x) \end{cases} \quad \text{if they exist}$$

$$E\left[\sum_{i=1}^{m} g(X_i)\right] = \sum_{i=1}^{m} E[g(X_i)]$$