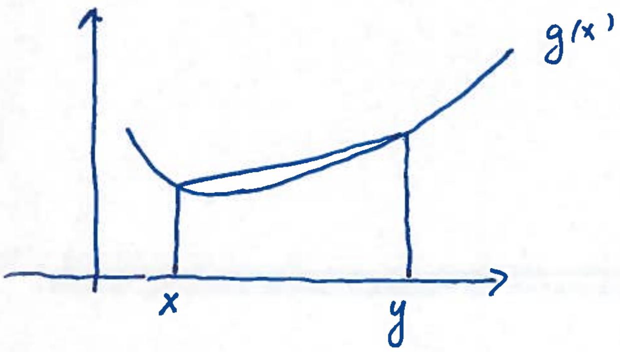
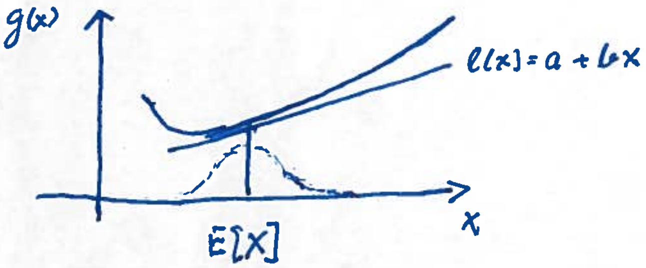
Definition

$g(x)$ is convex if $g(\lambda x + (1-\lambda) y) \le \lambda g(x) + (1-\lambda) g(y)$



Theorem 4.7.7 Jensen's Ulikhet

X random variable, $g(x)$ convex. Then $E[g(X)] \ge g(E[X])$



Proof.

Let $l(x) = a + bx$ be the tangent line in $(E[X], g[E[X]])$

Then $g(x) \ge a + bx, \quad \forall x$

and

$$E[g(X)] \ge a + b\,E[X] = l(E[X]) = g(E[X])$$

$$\left.\begin{array}{l} g(x) = x^2 \\ g(x) = \dfrac{1}{x}, \ x > 0 \end{array}\right\} \text{ both convex} \Rightarrow \left\{\begin{array}{l} E[X^2] \ge (E[X])^2 \\ E\left[\dfrac{1}{X}\right] \ge \dfrac{1}{E[X]} \end{array}\right.$$

$g(x)$ concave $\Rightarrow E[g(X)] \le g[E[X]]$

Chapter 5. Properties of a random sample

Random sample. Definition 5.1

$X_1, \dots, X_n$ random sample if $X_1, \dots, X_n$ are mutually independent with the same pmf or pdf.

Definition 5.2.1

$X_1, \dots, X_n$ random sample and let $T(x_1, \dots, x_n)$ be a real-valued or vector-valued function which domain includes the sample space of $(X_1, \dots, X_n)$. Then $Y = T(X_1, \dots, X_n)$ is called a statistic. The distribution of Y is called the sampling distribution of Y.

Examples. $$Y = \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

$$Y = S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 .$$

Theorem 5.2.6

$X_1, \dots, X_n$ random sample. $E[X_i] = \mu$. $Var[X_i] = \sigma^2 < \infty$ $i = 1, 2, \dots, n$. Then $E[\bar{X}] = \mu$, $Var[\bar{X}] = \frac{\sigma^2}{n}$ and $E[S^2] = \sigma^2$

Proof. $E[S^2] = \sigma^2$

$$E[S^2] = E\left[\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2\right] = E\left[\frac{1}{n-1} \sum_{i=1}^{n} (X_i^2 - 2X_i \bar{X} + \bar{X}^2)^2\right]$$

$$= E\left[\frac{1}{n-1}\left(\sum_{i=1}^{n} X_i^2 - 2n\bar{X} + n\bar{X}^2\right)\right] = \frac{1}{n-1}\left[n E[X^2] - n E[\bar{X}^2]\right]$$

$$= \frac{1}{n-1}\left(n(\sigma^2+\mu^2) - n\left(\frac{\sigma^2}{n}+\mu^2\right)\right) = \frac{\sigma^2(n-1)}{n-1} = \sigma^2$$

Assume $Y = \sum_{i=1}^{n} X_i$ has pdf $f_Y(y)$

Then $P(\bar{X} \le x) = P\left(\frac{Y}{n} \le x\right) = P(Y \le nx) = F_Y(nx)$

and $f_{\bar{X}}(x) = n f_Y(nx)$

$$M_{\bar{X}}(t) = E[e^{t\bar{X}}] = E\left[e^{t\frac{Y}{n}}\right] = M_Y\left(\frac{t}{n}\right) = \prod_{i=1}^{n} M_X\left(\frac{t}{n}\right) = \left(M_X\left(\frac{t}{n}\right)\right)^n$$

<u>Theorem 5.2.9</u>

X, Y independent with pdfs $f_X(x)$ and $f_Y(y)$, $Z = X+Y$

Then $f_Z(z) = \int_{-\infty}^{\infty} f_X(w) f_Y(z-w)\, dw$

<u>Proof.</u>

Let $\left.\begin{array}{l} Z = X+Y \\ W = X \end{array}\right\} \Rightarrow \begin{array}{l} Y = Z-W \\ X = W \end{array} \Rightarrow J = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$

and $f(z,w) = f_{X,Y}(w, z-w) = f_X(w) \cdot f_Y(z-w)$

$$\Rightarrow f_Z(z) = \int_{-\infty}^{\infty} f_X(w) f_Y(z-w)\, dw$$

This is the convolution formula.

__Example.__ U, V Cauchy distributed

$U \sim \text{Cauchy}(0, \sigma)$, $V \sim \text{Cauchy}(0, \tau)$ i.e.

$$f_U(u) = \frac{1}{\pi\sigma} \cdot \frac{1}{1+\left(\frac{u}{\sigma}\right)^2}, \quad f_V(v) = \frac{1}{\pi\tau} \cdot \frac{1}{1+\left(\frac{v}{\tau}\right)^2} \quad \begin{array}{l} -\infty < u < \infty \\ -\infty < v < \infty \end{array}$$

σ and τ are scale parameters.

$Z = U + V$. Exercise 5.7 $\Rightarrow$

$$f_Z(z) = \frac{1}{\pi(\sigma+\tau)} \cdot \frac{1}{\left(1+\left(\frac{z}{\sigma+\tau}\right)\right)^2}, \quad -\infty < z < \infty$$

$Z_1, \dots, Z_n$, iid Cauchy $(0,1) \Rightarrow \sum_{i=1}^{n} Z_i \sim \text{Cauchy}(0, n)$

and $f_{\frac{\Sigma Z_i}{n}}(z) = n f_{\Sigma Z_i}(nz) = n \cdot \frac{1}{\pi n} \cdot \frac{1}{1+\left(\frac{nz}{n}\right)^2} = \frac{1}{\pi} \cdot \frac{1}{1+z^2}$

i.e. Cauchy $(0,1)$.

5.3 Sampling from the normal distribution.

__Theorem 5.3.1__

$X_1, \dots, X_n$ random sample, $X_i \sim N(\mu, \sigma^2)$, $i = 1, 2, \dots, n$

Then

a) $\bar{X}$ and S^2 are independent random variables

b) $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

c) $(n-1)\frac{S^2}{\sigma^2} \sim \chi^2(n-1)$

Proof. Assume $\mu = 0$ and $\sigma^2 = 1$

a) $$S^2 = \frac{1}{m-1} \sum_{i=1}^{m} (X_i - \bar{X})^2 = \frac{1}{m-1}\left((X_1-\bar{X})^2 + \sum_{i=1}^{m}(X_i - \bar{X})^2\right)$$

Now $$\sum_{i=1}^{m}(X_i - \bar{X}) = 0 \Rightarrow X_1 - \bar{X} = -\sum_{i=2}^{m}(X_i - \bar{X})$$

$$\Rightarrow S^2 = \frac{1}{m-1}\left(\left(\sum_{i=2}^{m}(X_i-\bar{X})\right)^2 + \sum_{i=2}^{m}(X_i - \bar{X})^2\right)$$ which is a function of $X_2 - \bar{X}, \dots, X_m - \bar{X}$

We have $$f(x_1, \dots x_m) = \frac{1}{(2\pi)^{\frac{m}{2}}} e^{-\frac{1}{2}\sum_{i=1}^{m} x_i^2}$$

Let
$$\left.\begin{aligned} y_1 &= \bar{X} \\ y_2 &= X_2 - \bar{X} \\ &\vdots \\ y_m &= X_m - \bar{X} \end{aligned}\right\} \Rightarrow \begin{aligned} \bar{X} &= y_1 \Rightarrow X_1 = y_1 - \sum_{i=1}^{2} y_i \quad \text{since } \left(X_1 - y_1 = -\sum_{i=2}^{m}(X_i - \bar{X})\right) \\ X_2 &= y_2 + y_1 \\ &\vdots \\ X_m &= y_m + y_1 \end{aligned}$$

$$J = \begin{vmatrix} 1 & -1 & -1 & \cdots & -1 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 1 & 0 & \cdots & & 1 \end{vmatrix} = m$$

$$\Rightarrow f(y_1, \dots, y_m) = \frac{m}{(2\pi)^{\frac{m}{2}}} e^{-\frac{1}{2}\left(y_1 - \sum_{i=2}^{m} y_i\right)^2} \cdot e^{-\frac{1}{2}\sum_{i=2}^{m}(y_i + y_1)^2} \quad -\infty < y_i < \infty$$

Exponent: $$-\frac{1}{2}\left\{ \underline{y_1^2} - 2y_1 \sum_{i=2}^{m} y_i + \left(\sum_{i=2}^{m} y_i\right)^2 + \sum_{i=2}^{m} y_i^2 + 2y_1 \sum_{i=2}^{m} y_i + \underline{(m-1) y_1^2} \right\}$$

$$= -\frac{1}{2}\left\{ m y_1^2 + \left(\sum_{i=2}^{m} y_i\right)^2 + \sum_{i=2}^{m} y_i^2 \right\}$$

$$\Rightarrow f(y_1,\dots,y_n) = \frac{n^{\frac{1}{2}} e^{-\frac{1}{2} n y_1^2}}{2\pi^{\frac{1}{2}}} \cdot \frac{n^{\frac{1}{2}}}{(2\pi)^{\frac{n-1}{2}}} e^{-\frac{1}{2}\left[\sum_{i=2}^{n} y_i^2 + (\sum_{i=2}^{n} y_i)^2\right]}$$

$\Rightarrow Y_1$ and $Y_2,\dots,Y_n$ are independent and thereby

$\bar{X}$ and S.

<u>Proof of c</u>

$$(n-1)S_n^2 = \sum_{i=1}^{n}(X_i-\bar{X})^2 = \sum_{i=1}^{n-1}(X_i-\bar{X})^2 + (X_n-\bar{X})^2$$

$$= \sum_{i=1}^{n-1}(X_i - \bar{X}_{n-1} + \bar{X}_{n-1} - \bar{X})^2 + (X_n - \bar{X}_{n-1} + \bar{X}_{n-1} - \bar{X})^2$$

$$= \sum_{i=1}^{n-1}(X_i-\bar{X}_{n-1})^2 + \underbrace{(n-1)(\bar{X}_{n-1}-\bar{X})^2 + (X_n-\bar{X}_{n-1})^2 + (\bar{X}_{n-1}-\bar{X})^2}_{n(\bar{X}_{n-1}-\bar{X})^2}.$$

$$+ 2((X_n-\bar{X}_{n-1})(\bar{X}_{n-1}-\bar{X}))$$

We have

$$\bar{X}_{n-1} = \frac{n\bar{X}-X_n}{n-1} \Rightarrow n\bar{X}_{n-1} - \bar{X}_{n-1} = n\bar{X} - X_n$$

$$\Rightarrow \bar{X}_{n-1} - \bar{X} = \frac{\bar{X}_{n-1} - X_n}{n}$$

Thereby: $$(n-1)S_n^2 = \sum_{i=1}^{n-1}(X_i-\bar{X}_{n-1})^2 + \underbrace{\frac{(X_n-\bar{X}_{n-1})^2}{n} + (X_n-\bar{X}_{n-1})^2 - \frac{2(X_n-\bar{X}_{n-1})^2}{n}}_{\frac{n-1}{n}(X_n-\bar{X}_{n-1})^2}$$

$$= \sum_{i=1}^{n-1}(X_i-\bar{X}_{n-1})^2 + \frac{n-1}{n}(X_n-\bar{X}_{n-1})^2$$

$$= (n-2)S_{n-1}^2 + \frac{n-1}{n}(X_n-\bar{X}_{n-1})^2$$