$$\Rightarrow f(y_1, \dots, y_n) = \frac{n^{\frac{1}{2}} e^{-\frac{1}{2} n y_1^2}}{2\pi^{\frac{1}{2}}} \cdot \frac{n^{\frac{1}{2}}}{(2\pi)^{\frac{n-1}{2}}} e^{-\frac{1}{2}\left[\sum_{i=2}^{n} y_i^2 + \left(\sum_{i=2}^{n} y_i\right)^2\right]}$$

$\Rightarrow Y_1$ and $Y_2, \dots, Y_n$ are independent and thereby

$\bar{X}$ and S.

Proof of c

$$(n-1) S_n^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 = \sum_{i=1}^{n-1} (X_i - \bar{X})^2 + (X_n - \bar{X})^2$$

$$= \sum_{i=1}^{n-1} (X_i - \bar{X}_{n-1} + \bar{X}_{n-1} - \bar{X})^2 + (X_n - \bar{X}_{n-1} + \bar{X}_{n-1} - \bar{X})^2$$

$$= \sum_{i=1}^{n-1} (X_i - \bar{X}_{n-1})^2 + \underbrace{(n-1)(\bar{X}_{n-1} - \bar{X})^2 + (X_n - \bar{X}_{n-1})^2 + (\bar{X}_{n-1} - \bar{X})^2}_{n(\bar{X}_{n-1} - \bar{X})^2}.$$

$$+ 2\left((X_n - \bar{X}_{n-1})(\bar{X}_{n-1} - \bar{X})\right)$$

We have

$$\bar{X}_{n-1} = \frac{n\bar{X} - X_n}{n-1} \Rightarrow n\bar{X}_{n-1} - \bar{X}_{n-1} = n\bar{X} - X_n$$

$$\Rightarrow \bar{X}_{n-1} - \bar{X} = \frac{\bar{X}_{n-1} - X_n}{n}$$

Thereby: $$(n-1) S_n^2 = \sum_{i=1}^{n-1} (X_i - \bar{X}_{n-1})^2 + \underbrace{\frac{(X_n - \bar{X}_{n-1})^2}{n} + (X_n - \bar{X}_{n-1})^2 - \frac{2(X_n - \bar{X}_{n-1})^2}{n}}_{\frac{n-1}{n}(X_n - \bar{X}_{n-1})^2}$$

$$= \sum_{i=1}^{n-1} (X_i - \bar{X}_{n-1})^2 + \frac{n-1}{n}(X_n - \bar{X}_{n-1})^2$$

$$= (n-2) S_{n-1}^2 + \frac{n-1}{n}(X_n - \bar{X}_{n-1})^2$$

$n=2$

$$S_2^2 = \frac{1}{2}(X_2 - \bar{X}_1)^2 = Y^2, \quad \text{where } Y = \frac{X_2 - X_1}{\sqrt{2}}$$

$\Rightarrow \mathrm{Var}(Y) = 1 \Rightarrow Y \sim N(0,1) \Rightarrow Y^2 \sim \chi^2(1)$.

$n=3$.

$$2S_3^2 = S_2^2 + \frac{2}{3}(X_3 - \bar{X}_2)^2 = S_2^2 + \frac{2}{3}\left(X_3 - \frac{X_1}{2} - \frac{X_2}{2}\right)^2$$

$$Y = \frac{X_3 - \frac{X_1}{2} - \frac{X_2}{2}}{\sqrt{\frac{3}{2}}} \Rightarrow \mathrm{Var}[Y] = \frac{\frac{3}{2}}{\frac{3}{2}} = 1 \Rightarrow Y \sim N(0,1)$$

and $Y^2 \sim \chi^2(1) \Rightarrow 2S_3^2 = S_2^2 + Y^2 \sim \chi^2(1) + \chi^2(1) \sim \chi^2(2)$

since S_2^2 and $X_3 - \bar{X}_2$ are independent.

5.3.2. Student's t distribution and Snedecor's F

Let $X_1, \dots, X_n \sim N(\mu, \sigma^2)$ and independent, $E[\bar{X}] = \mu$

$$\mathrm{Var}[\bar{X}] = \frac{\sigma^2}{n} \quad \text{and} \quad \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

σ^2 unknown: $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ is an estimator for σ^2

and $S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}$ an estimator for σ

What is the distribution of $\frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}}$?

$$\frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} = \frac{\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}}{\frac{S}{\sigma}} = \frac{\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{S^2}{\sigma^2}}} \sim \frac{N(0,1)}{\sqrt{\frac{\chi^2(n-1)}{n-1}}} \quad \text{independent.}$$

Let $X \sim N(0,1)$ and $Y \sim \chi^2(n-1) = \text{gamma}\left(\frac{n-1}{2}, 2\right)$

Then $f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{(n-1)}{2}}} \cdot y^{\left(\frac{n-1}{2}\right)-1} e^{-\frac{y}{2}}, \quad -\infty < x < \infty, \quad 0 < y < \infty$

Define $\left.\begin{array}{l} Z = \dfrac{X}{\sqrt{\frac{Y}{n-1}}} \\ W = Y \end{array}\right\} \Rightarrow \begin{array}{l} X = \sqrt{\frac{W}{n-1}}\, Z \\ Y = W \end{array}$

$$\Rightarrow J = \begin{vmatrix} \sqrt{\frac{w}{n-1}} & \frac{\partial x}{\partial w} \\ 0 & 1 \end{vmatrix} = \sqrt{\frac{w}{n-1}}$$

$$f_{Z,W}(z,w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2 \frac{w}{n-1}} \cdot \frac{1}{\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-1}{2}}} w^{\left(\frac{n-1}{2}\right)-1} \cdot e^{-\frac{w}{2}} \cdot \sqrt{\frac{w}{n-1}} \qquad \begin{array}{l} -\infty < z < \infty \\ 0 < w < \infty \end{array}$$

$$\Rightarrow f_Z(z) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} \cdot \frac{1}{2^{\frac{n-1}{2}}} \cdot \frac{1}{\sqrt{n-1}} \int_0^\infty e^{-\frac{1}{2}\left(1+\frac{z^2}{n-1}\right)w} \cdot w^{\frac{n}{2}-1}\, dw$$

$$\underset{t=\frac{1}{2}\left(1+\frac{z^2}{n-1}\right)w}{=} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} \cdot \frac{1}{2^{\frac{n-1}{2}}} \cdot \frac{1}{\sqrt{n-1}} \int_0^\infty \frac{2^{\frac{n}{2}-1}\, t^{\frac{n}{2}-1} \cdot e^{-t}}{\left(1+\frac{z^2}{n-1}\right)^{\frac{n}{2}-1}} \cdot \frac{2}{1+\frac{z^2}{n-1}}\, dt$$

$$= \frac{1}{\sqrt{\pi(n-1)}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \cdot \frac{1}{\left(1+\frac{z^2}{n-1}\right)^{\frac{n}{2}}}$$

Definition 5.3.4

If T has a pdf given by $f_T(t) = \frac{1}{\sqrt{\pi p}} \cdot \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \cdot \frac{1}{\left(1+\frac{t^2}{p}\right)^{\frac{p+1}{2}}}$

$-\infty < t < \infty$, it is said to be t-distributed with p degrees of freedom.

Some properties.

$T = \dfrac{X}{\sqrt{\frac{Y}{p}}}$, X and Y are independent, $X \sim N(0,1)$, $Y \sim \chi^2(p)$

$p > 1$

$$E[T] = E[X] \cdot E\left[\frac{1}{\sqrt{\frac{X}{p}}}\right] = 0$$

$$Var[T] = E[X^2] \cdot E\left[\frac{p}{Y}\right] = p \cdot E\left[\frac{1}{Y}\right] \text{ where } Y \sim \Gamma\left(\tfrac{p}{2}, 2\right)$$

$$X \sim \text{gamma}(\alpha, \beta) \Rightarrow E[X^m] = \frac{\Gamma(\alpha + m)\beta^m}{\Gamma(\alpha)}, \quad m > -\alpha$$

$$E[X^{-1}] = \frac{\Gamma(\frac{p}{2} - 1)}{\Gamma(\frac{p}{2})} \cdot 2^{-1} = \frac{1}{2(\frac{p}{2} - 1)} = \frac{1}{p-2} \Rightarrow Var[T] = \frac{p}{p-2}, \; p > 2.$$

Definition 5.3.6

Let $X_1, \ldots, X_n \sim N(\mu_X, \sigma_X^2)$ and $Y_1, \ldots, Y_m \sim N(\mu_Y, \sigma_Y^2)$

Assume $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ are independent random samples. Then $F = \dfrac{\frac{S_X^2}{\sigma_X^2}}{\frac{S_Y^2}{\sigma_Y^2}} \sim$ Snedecor's F with $n-1$ and $m-1$ degrees of freedom.

$$F = \frac{\frac{\chi^2(n-1)}{n-1}}{\frac{\chi^2(m-1)}{m-1}}$$ independent.

Equivalently if $f_F(x) = \dfrac{\Gamma(\frac{p+q}{2})}{\Gamma(\frac{p}{2}) \cdot \Gamma(\frac{q}{2})} \cdot \left(\dfrac{p}{q}\right)^{\frac{p}{2}} \cdot \dfrac{x^{\frac{p}{2} - 1}}{(1 + \frac{p}{q}x)^{\frac{p+q}{2}}}, \; 0 < x < \infty$

then F is F (Fisher) distributed with p and q degrees of freedom.