Chapter 1. Probability Theory

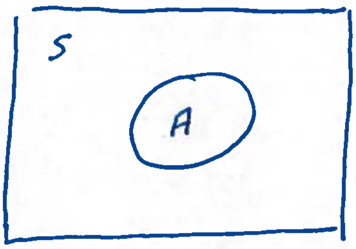
σ - algebra 1.2.1

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{B} \Rightarrow \left(\bigcup_{i=1}^{\infty} A_i^c \right)^c \in \mathcal{B} \Rightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{B}$$

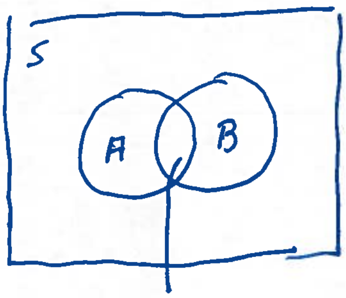
$$A_i = \left(-\frac{1}{i}, 1+\frac{1}{i}\right) \Rightarrow \bigcap_{i=1}^{\infty} A_i = [0,1]$$

$$A_i = \left[\frac{1}{i}, 1-\frac{1}{i}\right] \Rightarrow \bigcup_{i=1}^{\infty} A_i = (0,1)$$

Conditional Probability

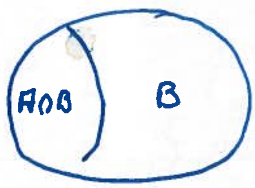


$$P(A) = \frac{P(A \cap S)}{P(S)} = P(A|S)$$



$A \cap B$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



Independence

$A_1, A_2, \ldots, A_n$ mutually independent if for any subcollection $A_{i_1}, \ldots, A_{i_k}$ we have:

$$P\left(\bigcap_{j=1}^{k} A_{i_j}\right) = \prod_{j=1}^{k} P(A_{i_j})$$

A and B independent $\Rightarrow$ $P(A \cap B) = P(A) \cdot P(B)$

$$\Rightarrow \quad P(A|B) = \frac{P(A) \cdot P(B)}{P(B)} = P(A)$$

A and B independent $\Rightarrow$ A^c and B, A and B^c; A^c and B^c are all independent.

Proof: $P(A \cap B^c) = P(A) - P(A \cap B) = P(A) - P(A) \cdot P(B)$

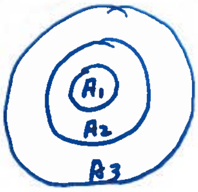
$= P(A)(1 - P(B)) = P(A) \cdot P(B^c)$

Multiplication rule

$$P(A \cap B \cap C) = P(A | B \cap C) \cdot P(B | C) \cdot P(C)$$

Convergence of sets

Let $A_1, A_2, \dots, A_n$ be events such that either $\{A_n\} \uparrow A$ or $\{A_n\} \downarrow A$. Then $\lim_{n \to \infty} P(A_n) = P(A)$



$$A = A_1 \cup (A_2 \cap A_1^c) \cup (A_3 \cap A_2^c) \cup \dots$$

$$P(A) = P(A_1) + P(A_2 \cap A_1^c) + \dots + P(A_n \cap A_{n-1}^c) + \dots$$

$$P(A) = \lim_{n \to \infty} B(\underbrace{P(A_1) + P(A_2 \cap A_1^c)}_{P(A_2)} + \dots + P(A_n \cap A_{n-1}^c)) = \lim_{n \to \infty} P(A_n)$$

$\{A_n\} \downarrow A$ Let $B_n = A_n^c$, $B = A^c$ $B_n \uparrow B$

$$\lim_{n \to \infty} P(B_n) = \lim_{n \to \infty} (1 - P(A_n)) = 1 - \lim_{n \to \infty} P(A_n) = P(B) = 1 - P(A)$$

$$\Rightarrow \lim_{n \to \infty} P(A_n) = P(A)$$

Theorem 1.5.3

Let $F_X(x)$ be the cdf of a random variable X. We have

a)
$0 \le F_X(x) \le 1$ since $F_X(x) = P(X \le x)$ is a probability.

b) $F_X(x) \le F_X(y)$ whenever $x \le y$ since if $A = \{s, X(s) \le x\}$ and $B = \{s : X(s) \le y\}$, then $A \subset B$

c) $\lim_{x \to \infty} F_X(x) = 1$

Proof: $A_n = \{s : X(s) \le n\}$, $A_n \uparrow S \Rightarrow \lim_{n \to \infty} P(A_n) = \lim_{n \to \infty} F_X(n) = \lim_{x \to \infty} F_X(x)$

$= P(S) = 1$

d) $\lim_{x \to -\infty} F_X(x) = 0$

Proof: Let $B_n = \{s : X(s) \le -n\}$. $B_n \downarrow \phi$

$\Rightarrow \lim_{n \to \infty} P(B_n) = \lim_{x \to -\infty} F_X(x) = P(\phi) = 0$

e) $F_X(x)$ is right continuous so: $F(x^+) = F(x)$

Proof. $B_n = \{s : X(s) \le x + \frac{1}{n}\}$, $B_n \downarrow \{s : X(s) \le x\} = B$

$\Rightarrow \lim_{n \to \infty} F(x + \frac{1}{n}) = \lim_{n \to \infty} P(B_n) = P(B) = P(s : X(s) \le x) = F_X(x)$

Identically distributed random variables

3 tosses with a coin

X = number of heads , Y = number of tails

X is always different from Y, but their distribution is the same.

Counting (Combinatorics)

Uniform probability models. $P(A) = \frac{g}{m} = \frac{\text{number of favourable}}{\text{number of possible}}$

Possible methods of counting

We want to draw r items from n distinct ones.

	Without replacement	With replacement
Ordered	$n!/(n-r)!$	n^r
Unordered	$\binom{n}{r}$	$\binom{n+r-1}{r}$ ←

An example

We want to draw 2 numbers from $\{1, 2, 3\}$ with replacement, and we do not care about the ordering

3 numbers ⇒ 3 bins and four walls

$|\;\;|\;\;|MM|$

$\;1\;\;2\;\;3$

How many ways can we put 2 markers in the bins.

Enough with two walls. The two at the end can be removed

M I I M   I M M I   M M I I

1 3   2 2   1 1

Two walls $(n-1)$ and two numbers can be arranged in $(3-1+2)!$ $\left((n-1+r)!\right)$ ways.

To remove double counting we get.

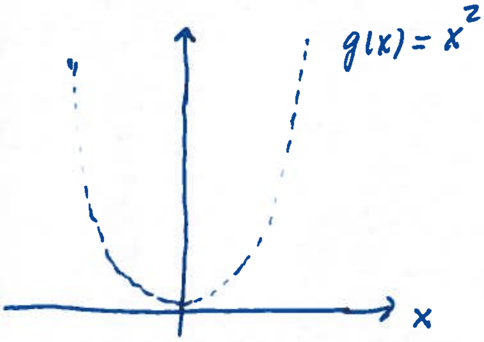
$$\frac{4!}{2!\,2!} \quad \text{or} \quad \frac{(n-1+r)!}{(n-1)!\,r!} = \binom{n-1+r}{r}$$

Chapter 2. Transformations and Expectation

$$Y = g(X) : \qquad g(x) : \mathcal{X} \to \mathcal{Y}.$$

$$F_Y(y) = P(g(X) \le y) = P(x \in \mathcal{X} : g(x) \le y) \overset{\text{cont.}}{=} \int_{\{x : g(x) \le y\}} f_X(x)\,dx$$

Example 2.1.7



X continuous with support set $\mathcal{X} = (-\infty, \infty)$

$$Y = g(X) = X^2$$

$$F_Y(y) = P(X^2 \le y) = P(-\sqrt{y} \le x \le \sqrt{y})$$

$$= P(X \le \sqrt{y}) - P(X \le -\sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$\Rightarrow f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})$$