$$F = \frac{\frac{\chi^2(p)}{p}}{\frac{\chi^2(q)}{q}} = \frac{\chi^2(p)}{\chi^2(q)} \frac{q}{p}$$

$$E[F] = \frac{q}{p} E[\chi^2(p)] \cdot E\left[\frac{1}{\chi^2(q)}\right] = \frac{q}{p} \cdot p \cdot \frac{1}{q-2} = \frac{q}{q-2}.$$

$$\chi^2(q) \sim \Gamma\left(\frac{q}{2}, 2\right) \Rightarrow E\left[\chi^2(q)^{-1}\right] = \frac{\Gamma\left(\frac{q}{2}-1\right)}{2\,\Gamma\left(\frac{q}{2}\right)} = \frac{1}{2\left(\frac{q}{2}-1\right)}$$

$$Var[F] = \frac{q^2}{p^2} E[(\chi^2(p))^2] \cdot E\left[\frac{1}{(\chi^2(q))^2}\right] - \left(\frac{q}{q-2}\right)^2$$

$$= \frac{q^2}{p^2}\left[\frac{p^2+2p}{(q-2)(q-4)}\right] - \frac{q^2}{(q-2)^2}. = \frac{2q^2(q+p-2)}{p(q-2)^2(q-4)}$$

$$E[X^{-2}] = \frac{\Gamma\left(\frac{q}{2}-1\right)}{2^2\Gamma\left(\frac{q}{2}\right)} = \frac{1}{2^2\left(\frac{q}{2}-1\right)\left(\frac{q}{2}-2\right)}$$

<u>Theorem 5.3.8</u>

a) $X \sim F_{p,q} \Rightarrow \frac{1}{X} \sim F_{q,p}$

b) $X \sim t_q \Rightarrow X^2 \sim F_{1,q}$

c) $X \sim F_{p,q} \Rightarrow \frac{\left(\frac{p}{q}\right)X}{1+\frac{p}{q}X} \sim \text{beta}\left(\frac{p}{2}, \frac{q}{2}\right)$

5.5 Convergence for random variables

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables defined by

X_n	0	n
$P(X=X_n)$	$1-\frac{1}{n}$	$\frac{1}{n}$

$E[X_n] = n \cdot \frac{1}{n} = 1, \quad \forall n \implies \lim_{n\to\infty} E[X_n] = 1$

$Var[X_n] = E[X_n^2] - [E[X_n]]^2 = n^2 \cdot \frac{1}{n} - 1 = n-1 \to \infty$

as $n \to \infty$

Given $\varepsilon > 0$

$$P(|X_n|<\varepsilon) = P(X_n<\varepsilon) = \begin{cases} 1-\frac{1}{n}, & (n>\varepsilon) \to 1 \text{ when } n\to\infty \\ 1, & n\le\varepsilon \end{cases}$$

$$\implies \lim_{n\to\infty} P(|X_n|<\varepsilon) = 1, \qquad X_n \xrightarrow{P} 0$$

Distribution

$$F_{X_n}(x) = \begin{cases} 0, & x<0 \\ 1-\frac{1}{n}, & 0\le x<n \\ 1, & x\ge n \end{cases} \implies \lim_{n\to\infty} F_{X_n}(x) = \begin{cases} 0, & x<0 \\ 1, & x\ge 0 \end{cases}$$

$$X_n \xrightarrow{D} 0$$

<u>Definition 5.5.1</u>

A sequence $\{X_n\}_{n=1}^{\infty}$ converges in probability to a random ~~variabl~~ variable X if $\forall \varepsilon > 0$

$$\lim_{n\to\infty} P(|X_n - X| \ge \varepsilon) = 0 \iff \lim_{n\to\infty} P(|X_n - X| < \varepsilon) = 1$$

Sometimes X is a constant.

Theorem 5.5.2 (Weak Law of large numbers)

Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. with $E[X_i]=\mu$ and $Var[X_i]=\sigma^2<\infty$

Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then $\forall \varepsilon > 0$, $\lim_{n\to\infty} P(|\bar{X}_n-\mu|<\varepsilon) = 1$

i.e. $\bar{X}_n \xrightarrow{P} \mu$.

Proof. $P(|\bar{X}_n-\mu|\geq\varepsilon) = P((\bar{X}_n-\mu)^2\geq\varepsilon^2) \leq \frac{E(\bar{X}_n-\mu)^2}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2}$

$\to 0$ as $n\to\infty$.

Example. $S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2$, $\quad S_n^2 \xrightarrow{P} \sigma^2$?

$P(|S_n^2-\sigma^2|\geq\varepsilon) = P((S_n^2-\sigma^2)^2\geq\varepsilon^2) \leq \frac{E[(S_n^2-\sigma^2)^2]}{\sigma^2} = \frac{Var\, S_n^2}{\varepsilon^2}$

$Var\, S_n^2 \to 0 \implies S_n^2 \xrightarrow{P} \sigma^2$.

Theorem 5.5.4

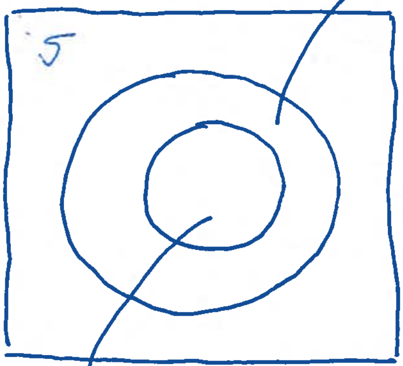
Suppose $\{X_n\}_{n=1}^{\infty}$ converges in probability to X and let h be continuous. Then $\{h(X_n)\}_{n=1}^{\infty}$ converges in probability to $h(X)$.

h continuous $\implies$ Given $\varepsilon>0$ there exists a δ such that

$|X_n - X| < \delta \implies |h(X_n)-h(X)|<\varepsilon$

Proof.

$\{w : |h(X_n) - h(X)| < \varepsilon\}$



$\{w : |X_n - X| < \delta\}$

$\forall\, \varepsilon > 0$ we have

$$\lim_{n\to\infty} P(|h(X_n) - h(x)| < \varepsilon) \geq \lim_{n\to\infty} P(|X_n - x| < \delta)$$

$\xrightarrow{n\to\infty} 1$ no matter what δ is.

Hence $\{X_n\}_{n=1}^{\infty} \xrightarrow{P} a \Rightarrow \{X_n^2\}_{n=1}^{\infty} \xrightarrow{P} a^2$, $\left\{\frac{1}{X_n}\right\}_{n=1}^{\infty} \xrightarrow{P} \frac{1}{a}$

$\{\sqrt{X_n}\}_{n=1}^{\infty} \xrightarrow{P} \sqrt{a}$

5.5.3 Convergence in Distributions

$\{X_n\}$ converges in distribution to a random variable X if $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$, $\forall x$ where $F_X(x)$ is continuous

Write $X_n \xrightarrow{D} X$

~~5.5.~~ Example. $X_1, \dots, X_n$ uniform $(0,1)$ and independent

$X_{(n)} = \max_{1 \leq i \leq n} X_i \Rightarrow F_{X_n}(x) = (F_X(x))^n = x^n, \quad 0 \leq x \leq 1$

$P(|X_{(n)} - 1| \geq \varepsilon) = P(X_{(n)} \geq 1 + \varepsilon) + P(X_{(n)} \leq 1 - \varepsilon)$

$= P(X_{(n)} \leq 1 - \varepsilon) = (1-\varepsilon)^n \xrightarrow[n\to\infty]{} 0 \Rightarrow X_{(n)} \xrightarrow{P} 1$