Now let $\varepsilon = \frac{t}{n}$

$$P\left(X_{(n)} \le 1 - \frac{t}{n}\right) = \left(1 - \frac{t}{n}\right)^n \xrightarrow[n \to \infty]{} e^{-t}$$

$$\Leftrightarrow P(n(X_{(n)}-1) \le -t) \xrightarrow[n\to\infty]{} e^{-t} \Rightarrow P(n(1-X_{(n)}) \le t) \xrightarrow[n\to\infty]{} 1 - e^{-t}$$

or $n(1-X_{(n)}) \xrightarrow{D}$ exponential (1)

<u>Theorem 5.5.12</u>

Assume $\{X_n\} \xrightarrow{P} X$. Then $\{X_n\}_{n=1}^{\infty} \xrightarrow{D} X$ or

$\lim\limits_{n\to\infty} F_{X_n}(x) = F_X(x)$, $\forall x$ where $F_X(x)$ is continuous

<u>Proof</u>. Let X be a point where $F_X(x)$ is continuous.

$\forall \varepsilon > 0$ and $\forall n$ we have.

$$F_{X_n}(x) = P(X_n \le x) = P(X_n \le x \cap (|X_n - x| < \varepsilon)) + P(X_n \le x \cap (|X_n - x| \ge \varepsilon))$$

$$\le P(X \le x + \varepsilon) + P(|X_n - x| \ge \varepsilon)$$

$$= F_X(x+\varepsilon) + P(|X_n - x| \ge \varepsilon)$$

Also $$F_X(x-\varepsilon) = P(X \le x - \varepsilon) = P(X \le x - \varepsilon \cap (|X_n - X| < \varepsilon))$$

$$+ P(X \le x - \varepsilon \cap (|X_n - x| \ge \varepsilon))$$

$$\le F_{X_n}(x) + P(|X_n - x| \ge \varepsilon)$$

Thereby $$F_X(x-\varepsilon) - P(|X_n - X| \ge \varepsilon) \le F_{X_n}(x) \le F_X(x+\varepsilon) + P(|X_n - x| \ge \varepsilon)$$

lim sup and lim inf.

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence

Let $b_n = \sup\{a_n, a_{n+1}, \dots\}$, $n = 1, 2, \dots$

and $c_n = \inf\{a_n, a_{n+1}, \dots\}$, -" —

$\{b_n\}$ is nonincreasing, $\{c_n\}$ is nondecreasing.

Thereby $\lim_{n\to\infty} b_n$ exist (maybe $-\infty$) is called lim sup

and $\lim_{n\to\infty} c_n$ exist (maybe ∞), is called lim inf.

Also $c_n \le a_n \le b_n$ Sandwich property

Let $b_n = \sup_{m\ge 1}\{F_{X_m}(x), F_{X_{m+1}}(x), \dots\} \le F_X(x+\varepsilon) + \sup_{m\ge 1}\{P(|X_m - x| \ge \varepsilon),$

$P(|X_{m+1} - x| \ge \varepsilon), \dots\}$

$\Rightarrow$ $\limsup F_{X_m}(x) \le F_X(x+\varepsilon)$

Let $c_n = \inf_{m\ge 1}\{F_{X_m}(x), F_{X_{m+1}}(x), \dots\} \ge F_X(x-\varepsilon) - \inf_{m\ge 1}\{P(|X_m - x| \ge \varepsilon)$

$P(|X_{m+1} - x| \ge \varepsilon \dots\}$

Thereby $\liminf_{n\to\infty} F_{X_m}(x) \ge F_X(x-\varepsilon)$

Thus we have $F_X(x-\varepsilon) \le \lim_{n\to\infty} F_{X_m}(x) \le F_X(x+\varepsilon)$

Since $F_X(x)$ is continuous in x, we have $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$

Important $\{X_n\}_{n=1}^{\infty} \xrightarrow{D} X \not\Rightarrow \{X_n\}_{n=1}^{\infty} \xrightarrow{P} X$

<u>Example</u>

Let $X \sim N(0,1)$ and define $X_n = \begin{cases} X & \text{if } n \text{ is even} \\ -X & \text{if } n \text{ is odd} \end{cases}$

$\forall \varepsilon > 0$, we have

$$P(|X_n - X| \geq \varepsilon) = \begin{cases} 0 & \text{for } n \text{ even} \\ 2(1-\Phi(\frac{\varepsilon}{2})) & \text{for } n \text{ odd.} \end{cases}$$

<u>Theorem 5.5.13</u>

$$\{X_n\}_{n=1}^{\infty} \xrightarrow{P} \mu \iff \{X_n\}_{n=1}^{\infty} \xrightarrow{D} \mu$$

<u>Proof</u>. $\{X_n\}_{n=1}^{\infty} \xrightarrow{D} \mu \Rightarrow P(X_n \leq x) = F_{X_n}(x) \to \begin{cases} 0, & x \leq \mu \\ 1, & x \geq \mu \end{cases}$

Then

$\forall \varepsilon > 0 \quad P(|X_n - \mu| < \varepsilon) \leq F_{X_n}(\mu+\varepsilon) - F_{X_n}(\mu-\varepsilon) \xrightarrow[n\to\infty]{} 1$

<u>Theorem 5.5.15</u> Central limit theorem

$\{X_n\}_{n=1}^{\infty}$ i.i.d with $E[X_i] = \mu$ and $0 < Var[X_i] = \sigma^2 < \infty$

Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and let $F_{X_n}(x)$ be the cdf of.

$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}}$. $\forall x \in R$ we have $\lim_{n\to\infty} F_{X_n}(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$.

Since $F_X(x)$ is continuous in x, we have $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$.

Important $\{X_n\}_{n=1}^{\infty} \xrightarrow{D} X \not\Rightarrow \{X_n\}_{n=1}^{\infty} \xrightarrow{P} X$

Example

Let $X \sim N(0,1)$ and define $X_n = \begin{cases} X & \text{if } n \text{ is even} \\ -X & \text{if } n \text{ is odd} \end{cases}$

$\forall \varepsilon > 0$, we have

$$P(|X_n - X| \geq \varepsilon) = \begin{cases} 0 & \text{for } n \text{ even} \\ 2(1-\Phi(\frac{\varepsilon}{2})) & \text{for } n \text{ odd.} \end{cases}$$

Theorem 5.5.13

$$\{X_n\}_{n=1}^{\infty} \xrightarrow{P} \mu \iff \{X_n\}_{n=1}^{\infty} \xrightarrow{D} \mu$$

Proof $\Leftarrow$

$$\{X_n\}_{n=1}^{\infty} \xrightarrow{D} \mu \Rightarrow P(X_n \leq x) = F_{X_n}(x) \longrightarrow \begin{cases} 0, & x < \mu \\ 1, & x \geq \mu \end{cases}$$

$\forall \varepsilon > 0$ $P(|X_n - \mu| < \varepsilon) = P(X_n < \mu + \varepsilon) - P(X_n \leq \mu - \varepsilon)$

$$= P(X_n \leq \mu+\varepsilon) - P(X_n = \mu+\varepsilon) - P(X_n \leq \mu - \varepsilon)$$

$$= \underbrace{F_{X_n}(\mu+\varepsilon)}_{\xrightarrow{n\to\infty} 1} - \underbrace{P(X_n = \mu+\varepsilon)}_{\xrightarrow{n\to\infty} 0} - \underbrace{F_{X_n}(\mu-\varepsilon)}_{\xrightarrow{n\to\infty} 0} \xrightarrow{n\to\infty} 1$$

Theorem 5.5.15. Central limit theorem

$\{X_n\}_{n=1}^{\infty}$ iid with $E[X_i] = \mu$ and $0 < Var[X_i] = \sigma^2 < \infty$

Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and let $F_{X_n}(x)$ be the cdf of

$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}}$. $\forall x \in R$ we have $\lim_{n\to\infty} F_{X_n}(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$

Proof. Suppose $M_{X_i}(t)$ exists for $|t| \leq h$.

Then $\dfrac{\bar{X}_n - \mu}{\sigma} = \dfrac{1}{n} \sum_{i=1}^{n} \left(\dfrac{X_i - \mu}{\sigma} \right)$ and $\dfrac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = \dfrac{1}{\sqrt{n}} \sum_{i=1}^{n} \Big(\underbrace{\dfrac{X_i - \mu}{\sigma}}_{= Y_i} \Big)$

$$M_{\sum_{i=1}^{n} Y_i} = (M_Y(t))^n \quad \text{and} \quad M_{\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}}(t) = \left(M_Y\left(\tfrac{t}{\sqrt{n}}\right) \right)^n$$

A Taylor expansion around 0 gives:

$$M_Y\left(\tfrac{t}{\sqrt{n}}\right) = M_Y(0) + M_Y'(0) \cdot \frac{t}{\sqrt{n}} + \frac{M_Y''(0) \cdot \left(\frac{t}{\sqrt{n}}\right)^2}{2} + R_Y\left(\tfrac{t}{\sqrt{n}}\right)$$

$$= 1 + \frac{\left(\frac{t}{\sqrt{n}}\right)^2}{2} + R_Y\left(\tfrac{t}{\sqrt{n}}\right)$$

and $\displaystyle\lim_{n \to \infty} \frac{R_Y\left(\frac{t}{\sqrt{n}}\right)}{\left(\frac{t}{\sqrt{n}}\right)^2} = 0$ for fixed t

For fixed t this implies $\displaystyle\lim_{n \to \infty} \frac{R_Y\left(\frac{t}{\sqrt{n}}\right)}{\left(\frac{1}{\sqrt{n}}\right)^2} = \lim_{n \to \infty} n R_Y\left(\tfrac{t}{\sqrt{n}}\right) = 0$

and

$$\lim_{n \to \infty} \left[1 + \frac{1}{n} \left(\frac{t^2}{2} + n R_Y\left(\tfrac{t}{\sqrt{n}}\right) \right) \right]^n = e^{\frac{t^2}{2}} = M_Z(t)$$

since $\displaystyle\lim_{n \to \infty} \left(1 + \frac{a_n}{n}\right)^n = e^a$ if $\{a_n\} \xrightarrow[n \to \infty]{} a$.

where $Z \sim N(0,1) \implies \dfrac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{D} Z$.

Theorem 5.5.17 Slutsky's Theorem

$\{X_n\}_{n=1}^{\infty} \xrightarrow{D} X$ and $\{Y_n\}_{n=1}^{\infty} \xrightarrow{P} a$. Then

a) $\{X_n Y_n\}_{n=1}^{\infty} \xrightarrow{D} aX$

b) $(X_n + Y_n)_{n=1}^{\infty} \xrightarrow{D} X + a$

Example

$$T_n = \frac{\bar{X}_n - \mu}{\frac{S_n}{\sqrt{n}}} = \frac{\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}}{\frac{S_n}{\sigma}} = \underbrace{\frac{\sigma}{S_n}}_{\xrightarrow{P} 1} \underbrace{\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}}_{\xrightarrow{D} N(0,1)}$$

Since $S_n^2 \xrightarrow{P} \sigma^2 \Rightarrow S_n \xrightarrow{P} \sigma$

$$\Rightarrow \quad T_n \xrightarrow[n\to\infty]{D} Z \sim N(0,1)$$

5.5.4. The Delta method

$X \sim$ random variable. Assume $Y = \frac{1}{X} = g(X)$

$$g'(x) = -\frac{1}{x^2}$$

In general for differentiable functions

$$g(x) = g(a) + g'(a)(x-a) + \frac{g''(a)(x-a)^2}{2} + R(x,a)$$

Therefore a natural approximation

$$Y = \frac{1}{\mu} + \left(-\frac{1}{\mu^2}(X-\mu) + R\right) \Rightarrow E[Y] \approx \frac{1}{\mu} \text{ and}$$

$$Var[Y] \approx \frac{1}{\mu^4} Var[X]$$