Negative binomial distribution

Let X be the number of trials until the r-th success

$P(X=x \mid r, p) = P($A occurs $r-1$ times in $x-1$ trials and A occurs in the x-th trial$)$

$$= \binom{x-1}{r-1} p^{r-1} (1-p)^{x-r} p \quad , \quad x = r, r+1, \dots$$

$$= \binom{x-1}{r-1} p^{r} (1-p)^{x-r} \quad , \quad x = r, r+1, \dots$$

$r = 1$ gives

$$P(X = x \mid 1, p) = p(1-p)^{x-1}, \quad x = 1, 2, \dots$$

which is the geometric distribution, for which

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} p(1-p)^{x-1} \overset{s=x-1}{=} pe^t \sum_{s=0}^{\infty} (e^t(1-p))^s = \frac{pe^t}{1-e^t(1-p)}$$

Need $e^t(1-p) < 1 \iff e^t < \frac{1}{1-p}$ or $t < \ln\left(\frac{1}{1-p}\right)$

$$M_X'(t) = \frac{pe^t}{1-e^t(1-p)} + \frac{pe^t e^t(1-p)}{(1-e^t(1-p))^2} = \frac{pe^t}{1-e^t(1-p)}\left[1 + \frac{e^t(1-p)}{1-e^t(1-p)}\right]$$

$$= \frac{pe^t}{(1-e^t(1-p))^2} . \qquad M_X'(0) = \frac{1}{p} = E[X]$$

Similarly: $Var[X] = E[X^2] - (E[X])^2 = \frac{1-p}{p^2}.$

Let $Y_1 \sim$ the number of trials until the 1. success

$\vdots$

$Y_i \sim$ — " — between the $(i-1)$-th and the i-th success.

$X \sim$ negative binomial $\Rightarrow$ $X = \sum_{i=1}^{r} Y_i$ where $Y_1, \dots, Y_r$ are identical, independent geometric distributed.

Thereby, $E[X] = \frac{r}{p}$, $Var[X] = \frac{r(1-p)}{p^2}$

and $M_X(t) = \left[\frac{pe^t}{1-(1-p)e^t}\right]^r$

For X negative binomial let $Y = X - r$

$\Rightarrow P(Y=y) = P(X-r=y) = P(X=r+y)$

$= \binom{r+y-1}{r-1} p^r (1-p)^y, \quad y = 0, 1, \dots$

Bernoulli, Binomial, Geometric, Negative binomial

The Gamma distribution

The gammafunction is defined as

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$$

$$\Gamma(\alpha+1) = \int_0^\infty t^\alpha e^{-t}\, dt = \left[-t^\alpha e^{-t}\right]_0^\infty + \int_0^\infty \alpha t^{\alpha-1} e^{-t}\, dt = \alpha \Gamma(\alpha)$$

$\Rightarrow \quad \Gamma(\alpha+1) = \alpha!$ if α is a positive integer.

$f(x) = \dfrac{x^{\alpha-1} e^{-x}}{\Gamma(\alpha)}, \quad x > 0, \ \alpha > 0$ is a pdf for a random variable X.

Let $Y = \beta X \Rightarrow X = \dfrac{Y}{\beta}$

$$f_Y(y) = \frac{y^{\alpha-1}}{\beta^{\alpha-1}} \frac{e^{-\frac{y}{\beta}}}{\Gamma(\alpha)} \cdot \frac{1}{\beta} = \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-\frac{y}{\beta}}, \quad y > 0, \ \alpha > 0, \ \beta > 0$$

This is defined as the pdf in the gamma distribution.

α is called the shape parameter and β the scale parameter.

Since $f_Y(y)$ is a density we have

$$\int_0^\infty y^{\alpha-1} e^{-\frac{y}{\beta}} = \Gamma(\alpha)\beta^\alpha$$

$$E[Y^m] = \int_0^\infty \frac{y^{\alpha+m-1} e^{-\frac{y}{\beta}}}{\Gamma(\alpha)\beta^\alpha} = \frac{\Gamma(\alpha+m)\beta^{\alpha+m}}{\Gamma(\alpha)\beta^\alpha}$$

provided $\alpha + m - 1 > 0$ or $m > 1 - \alpha$

That gives $E[X] = \dfrac{\Gamma(\alpha+1)\beta}{\Gamma(\alpha)} = \alpha\beta$

$$E[X^2] = \frac{\Gamma(\alpha+2)\beta^2}{\Gamma(\alpha)} = \alpha(\alpha+1)\beta^2$$

and $\quad Var[Y] = \alpha(\alpha+1)\beta^2 - \alpha^2\beta^2 = \alpha\beta^2$

<u>Useful results</u>

If $X \sim gamma(\alpha,\beta) \Rightarrow cX \; gamma(\alpha, c\beta)$

If $m > 1-\alpha \Rightarrow E[X^m]$ exist

$$M_X(t) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty e^{tx}x^{\alpha-1}e^{-\frac{x}{\beta}}dx = \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty x^{\alpha-1}e^{-\frac{x}{\beta/1-\beta t}}dt$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha}\Gamma(\alpha).\frac{\beta^\alpha}{(1-\beta t)^\alpha} = \frac{1}{(1-\beta t)^\alpha} \quad \text{provided } 1-\beta t > 0$$

or $t < \frac{1}{\beta}$.

$X \sim gamma(1,\beta) \Rightarrow f_X(x) = \dfrac{1}{\Gamma(1)}\cdot\dfrac{1}{\beta}e^{-\frac{x}{\beta}} = \dfrac{1}{\beta}e^{-\frac{x}{\beta}}, \; x>0, \beta>0$

i.e. $\quad X \sim exp(\frac{1}{\beta})$ and $M_X(t) = \dfrac{1}{1-\beta t}$

and $E[X] = \beta, \quad Var[X] = \beta^2$.

$X \sim gamma(\alpha_1,\beta)$, $Y \sim gamma(\alpha_2,\beta)$ and independent

Let $\quad Z = X+Y$

$$M_Z(t) = \frac{1}{(1-\beta t)^{\alpha_1}}\cdot\frac{1}{(1-\beta t)^{\alpha_2}} = \frac{1}{(1-\beta t)^{\alpha_1+\alpha_2}}$$

$$\Rightarrow Z \sim gamma(\alpha_1+\alpha_2,\beta)$$

Hence $\quad X_1,\dots,X_n \sim exp(\frac{1}{\beta})$ and independent

$$\Rightarrow Z = \sum_{i=1}^{n} X_i \sim gamma(n,\beta)$$

If $X \sim \text{gamma}(\frac{p}{2}, 2)$

$$\Rightarrow \quad f_X(x) = \frac{1}{\Gamma(\frac{p}{2})\, 2^{\frac{p}{2}}} x^{\frac{p}{2}-1} e^{-\frac{x}{2}}, \quad x > 0$$

This is the density in the χ^2 distribution with p degrees of freedom

<u>Example</u>. $X_1, \ldots, X_n$ independent and identically normally distributed with unknown expectation μ and unknown variance σ^2.

$$\hat{\sigma}^2 = S^2 = \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n-1}$$

Then $V = \dfrac{S^2(n-1)}{\sigma^2} \sim \chi^2(n-1) \sim \text{gamma}(\frac{(n-1)}{2}, 2)$

and $S^2 = \dfrac{\sigma^2}{n-1} V \sim \text{gamma}\left(\frac{n-1}{2}, \frac{2\sigma^2}{n-1}\right)$

<u>Poisson process</u>

$$X \sim Po(\lambda t) \Rightarrow P(X = x) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}, \quad x = 0, 1, 2, \ldots$$

Let T_1 be the time to the first event. $T_1 \sim \exp(\lambda)$

$$\Rightarrow f_{T_1}(t) = \lambda e^{-\lambda t} = \frac{1}{\beta} e^{-\frac{1}{\beta} t} \sim \Gamma(1, \beta), \quad \beta = \frac{1}{\lambda}, \quad \lambda > 0, \; t > 0.$$

Let T_α = time to event α $\Rightarrow$ $T_\alpha \sim \Gamma(\alpha, \beta)$, α integer

$P(T_\alpha \le t) \Leftrightarrow P(X \ge \alpha)$ where $X \sim Po(\lambda t)$.

The Beta distribution

$X \sim$ gamma$(\alpha, 1)$ and $Y \sim$ gamma$(\beta, 1)$

$$\Rightarrow \quad \frac{X}{X+Y} \sim \text{beta}(\alpha, \beta)$$

The pdf is given by

$$f(x|\alpha,\beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1, \ \alpha > 0, \ \beta > 0$$

where $B(\alpha,\beta)$ denotes the beta function.

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

For $m > -\alpha$ we get

$$E[x^m] = \frac{1}{B(\alpha,\beta)} \int_0^1 x^m x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^1 x^{\alpha+m-1}(1-x)^{\beta-1}\,dx = \frac{B(\alpha+m,\beta)}{B(\alpha,\beta)}$$

$$= \frac{\Gamma(\alpha+m)\,\Gamma(\beta)\cdot\Gamma(\alpha+\beta)}{\Gamma(\alpha+m+\beta)\cdot\Gamma(\alpha)\cdot\Gamma(\beta)} = \frac{\Gamma(\alpha+m)\cdot\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+m)\cdot\Gamma(\alpha)}$$

$$\Rightarrow E[x] = \frac{\Gamma(\alpha+1)\cdot\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1)\cdot\Gamma(\alpha)} = \frac{\alpha}{\alpha+\beta}$$

$$E[x^2] = \frac{\Gamma(\alpha+2)\cdot\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+2)\cdot\Gamma(\alpha)} = \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)}$$

and
$$\text{Var}[x] = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} - \frac{\alpha^2}{(\alpha+\beta)^2} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$