Chapter 8. Hypothesis Testing

Definition 8.1.1

A hypothesis is a statement about a population parameter

$$H_0 : \theta \in \Omega_0 \qquad H_1 : \theta \in \Omega_0^c$$

Definition 8.1.3

A hypothesis procedure is a rule that specifies

1. For which sample values the decision is to not reject H_0 and thereby no conclusion is reached.
2. For which sample values the decision is to reject H_0 and conclude with H_1.

Whether to reject or not is based on a statistic $T(\underline{X})$

8.2. Methods of finding tests.

Definition 8.2.1. The likelihood ratio test statistic (LRT) for testing $H_0: \theta \in \Omega_0$ against $H_1: \theta \in \Omega_0^c$ is

$$\lambda(\underline{x}) = \frac{\sup_{\theta \in \Omega_0} L(\theta | \underline{x})}{\sup_{\theta} L(\theta | \underline{x})} \quad \begin{array}{l} \text{restricted maximum likelihood} \\ \text{unrestricted maximum likelihood.} \end{array}$$

Thereby $\lambda(\underline{x}) = \dfrac{L(\hat{\theta}_0 | \underline{x})}{L(\hat{\theta} | \underline{x})}$ where $\hat{\theta}_0$ maximizes the restricted likelihood

$\hat{\theta}$ — " — unrestricted — " —

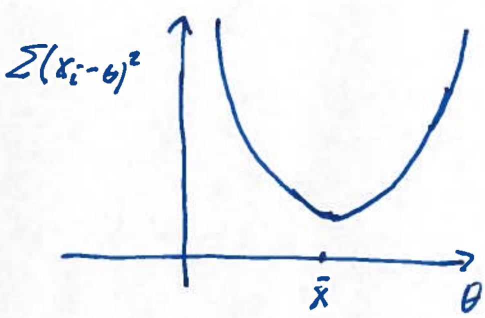
H_0 is rejected for small values of $\lambda(\underline{x})$, $\lambda(\underline{x}) \le c < 1$

<u>Example</u>

$X_1, \ldots, X_n$ iid $N(\theta, \sigma^2)$, σ^2 unknown

$H_0: \theta \le \theta_0$, $H_1: \theta > \theta_0$

$$\lambda(\underline{x}) = \frac{\sup\limits_{\theta \le \theta_0} (2\pi)^{-\frac{n}{2}} \sigma^{-n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2}}{\sup\limits_{\theta} (2\pi)^{-\frac{n}{2}} \sigma^{-n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2}}$$



$$= \begin{cases} 1 & \text{if } \theta_0 \ge \bar{x} \\ e^{-\frac{1}{2\sigma^2}\left\{\sum_{i=1}^{n}(x_i-\theta_0)^2 - \sum_{i=1}^{n}(x_i-\bar{x})^2\right\}}, & \theta_0 < \bar{x} \end{cases}$$

$$= \begin{cases} 1 & \text{if } \theta_0 \ge \bar{x} \\ e^{-\frac{1}{2\sigma^2} n(\bar{x}-\theta_0)^2}, & \theta_0 < \bar{x} \end{cases}$$

Reject if $-\frac{1}{2\sigma^2} n(\bar{x}-\theta_0)^2 < \log c \iff (\bar{x}-\theta_0)^2 > -\frac{2\sigma^2}{n}\log c$

or since $\bar{x} > \theta_0$, $\bar{x} - \theta_0 > \sqrt{-\frac{2\sigma^2}{n}\log c}$ or $\bar{x} > \theta_0 + \sqrt{-\frac{2\sigma^2}{n}\log c}$

<u>Theorem 8.2.4</u>

If $T(\underline{X})$ is a sufficient statistic for θ and $\lambda^*(t)$ and $\lambda(\underline{x})$ are the LRT statistic based on T and $\underline{X}$,

then $\lambda^*(T(\underline{x})) = \lambda(\underline{x})$, $\forall \underline{x}$

Proof.

We have $f(\underline{x}|\theta) = P(\underline{X}=\underline{x} \cap T(\underline{X}) = T(\underline{x})) = \underbrace{P(\underline{X}=\underline{x} \mid T(\underline{X}) = T(\underline{x}))}_{h(\underline{x})} \cdot \underbrace{P(T(\underline{X}) = T(\underline{x}))}_{g(T(\underline{x})|\theta)}$

Thereby
$$\lambda(\underline{x}) = \frac{\sup_{\theta\in\Omega_0} L(\theta|\underline{x})}{\sup_{\Omega} L(\theta|\underline{x})} = \frac{\sup_{\Omega_0} f(\underline{x}|\theta)}{\sup_{\Omega} f(\underline{x}|\theta)} = \frac{\sup_{\Omega_0} g(T(\underline{x})|\theta)\, h(\underline{x})}{\sup_{\Omega} g(T(\underline{x})|\theta)\, h(\underline{x})}$$

$$= \frac{\sup_{\Omega_0} g(T(\underline{x})|\theta)}{\sup_{\Omega} g(T(\underline{x})|\theta)} = \frac{\sup_{\Omega_0} L^{\circ}(\theta|T(\underline{x}))}{\sup_{\Omega} L^{\circ}(\theta|T(\underline{x}))} = \lambda^{\circ}(T(\underline{x}))$$

Example

$$f(x|\theta) = \begin{cases} e^{-(x-\theta)}, & x \ge \theta \\ 0, & x < \theta \end{cases} \Rightarrow F_X(x|\theta) = 1 - e^{-(x-\theta)}, \quad x \ge \theta$$

$X_1, \dots, X_n$ iid. $Y = \min_i X_i = T(\underline{X})$ is sufficient for θ.

$$F_Y(y|\theta) = 1 - (1 - F_X(y|\theta))^n = 1 - e^{-n(y-\theta)}, \quad y \ge \theta$$

and
$$f_Y(y|\theta) = \begin{cases} n e^{-n(y-\theta)}, & y \ge \theta, \\ 0, & y < \theta \end{cases}$$ increasing in θ.

$H_0: \theta \le \theta_0$, $H_1: \theta > \theta_0$

$$\lambda(\underline{y}) = \lambda(T(\underline{x})) = \frac{\sup_{\theta\le\theta_0} n e^{-n(y-\theta)}}{\sup_{\theta} n e^{-n(y-\theta)}} = \frac{\sup_{\theta\le\theta_0} n e^{-n(y-\theta)}}{n}$$

$$= \begin{cases} 1, & \theta_0 \ge y \\ e^{-n(y-\theta_0)}, & \theta_0 < y. \end{cases}$$

Reject H_0 if $\lambda(y) \le c \iff -n(y-\theta_0) \le \log c$

$\iff y \ge \theta_0 - \frac{\log c}{n}$.

Nuissance parameters

$X_1, \dots, X_n \sim$ iid $N(\mu, \sigma^2)$, σ^2 unknown.

$H_0: \mu \le \mu_0 \qquad H_1: \mu > \mu_0$

$$L(\mu, \sigma^2 | \underline{x}) = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$\log L(\mu, \sigma^2 | \underline{x}) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 - \frac{1}{2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\sigma^2}$ is maximized for $\mu = \bar{x}$

Given μ, $\sum_{i=1}^{n}(x_i-\mu)^2$ is a constant k and $\log L(\mu, \sigma^2 | \underline{x})$

is maximized for $\sigma^2 = \frac{k}{n} \Rightarrow \log L(\mu, \sigma^2 | \underline{x}) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\frac{k}{n} - \frac{n}{2}$

For $\mu \le \mu_0 < \bar{x}$, $\log L(\mu, \sigma^2 | \underline{x})$ is maximized for $\mu = \mu_0$

and $\sigma^2 = \hat{\sigma}_0^2 = \frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{n}$.

Hence.

$$\lambda(\underline{x}) = \frac{\sup_{\mu,\sigma^2:\ \mu\le\mu_0,\ \sigma^2>0} L(\mu,\sigma^2|\underline{x})}{\sup_{\mu,\sigma^2\ -\infty<\mu<\infty,\ \sigma^2>0} L(\mu,\sigma^2|\underline{x})} = \begin{cases} 1, & \mu_0 \ge \hat{\mu} = \bar{x} \\ \frac{L(\mu_0, \hat{\sigma}_0^2 | \underline{x})}{L(\hat{\mu}, \hat{\sigma}^2 | \underline{x})}, & \mu_0 < \bar{x} \end{cases}$$