8.3 Methods of Evaluating Tests.

Type of errors $\qquad H_0: \theta \in \Omega_0, \qquad H_1: \theta \in \Omega_0^c$

		Not reject	Reject.
Truth	H_0	Correct decision	Type 1 error
	H_1	Type 2 error	Correct decision

Let R be the rejection region for a test

$$P(X \in R \mid \theta) = \begin{cases} P(\text{Type 1 error}) & \text{if } \theta \in \Omega_0 \\ 1 - P(\text{Type 2 error}) & \text{if } \theta \in \Omega_0^c \end{cases}$$

<u>Definition 8.3.1</u>

The power function of a hypothesis test with rejection region R is the function $\beta(\theta) = P(X \in R \mid \theta)$, where ~~...~~ R is the rejection region.

<u>Example</u> $\quad X_1, \dots, X_n \sim N(\theta, \sigma^2)$, σ^2 known.

$H_0: \theta \le \theta_0, \qquad H_1: \theta > \theta_0$

Reject if $\dfrac{\bar{X} - \theta_0}{\frac{\sigma}{\sqrt{n}}} \ge c'$

$$\beta(\theta) = P\left(\frac{\bar{X} - \theta_0}{\frac{\sigma}{\sqrt{n}}} \ge c' \,\Big|\, \theta \right) = P\left(\frac{\bar{X} - \theta}{\frac{\sigma}{\sqrt{n}}} \ge c' + \frac{\theta_0 - \theta}{\frac{\sigma}{\sqrt{n}}} \right)$$

$N(0,1)$

$$= P\left(Z \ge c' + \frac{\theta_0 - \theta}{\frac{\sigma}{\sqrt{n}}} \right) \longrightarrow \begin{cases} 1 & \text{if } \theta \to \infty \\ 0 & \text{-- " --} \ \theta \to -\infty \\ = \alpha & \text{if } \theta = \theta_0 \text{ and } P(Z \ge c') = \alpha \end{cases}$$

Definition 8.3.5 and 8.3.6

For $\alpha \in (0,1)$, a test with $\sup_{\theta \in \Omega_0} \beta(\theta) = \alpha$ is a size α test

and a level α-test if $\sup_{\theta \in \Omega_0} \beta(\theta) \leq \alpha$.

In general a size α LRT is constructed by choosing

c such that $\sup_{\theta \in \Omega_0} P(\lambda(X) \leq c) = \alpha$

Example $f(x|\theta) = \begin{cases} e^{-(x-\theta)}, & x \geq \theta \\ 0, & x < \theta \end{cases}$ $H_0: \theta \leq \theta_0$, $H_1: \theta > \theta_0$

$Y = \min_i X_i$.

$\sup_{\theta \in \Omega_0} P(\lambda(Y) \leq c) = \sup_{\theta \leq \theta_0} P\left(Y \geq \theta_0 - \frac{\log c}{n}\right) = \alpha$

$\Leftrightarrow \sup_{\theta \leq \theta_0} e^{-n\left(\theta_0 - \frac{\log c}{n} - \theta\right)} = \alpha$

$\Leftrightarrow \sup_{\theta \leq \theta_0} e^{\log c - n(\theta_0 - \theta)} = \alpha \Leftrightarrow e^{\log c} = \alpha \rightarrow c = \alpha$

Hence Reject $H_0: \theta \leq \theta_0$ if $Y \geq \theta_0 - \frac{\log \alpha}{n}$.

8.3.2 Most Powerful test

Let C be a class of tests for testing $H_0: \theta \in \Omega_0$ against $H_1: \theta \in \Omega_0^c$. A test in class C with power function $\beta(\theta)$ is a uniformly most powerful (UMP) test if $\beta(\theta) \geq \beta'(\theta)$, $\forall \theta \in \Omega_0^c$ and for every $\beta'(\theta)$ that is a power function of a test in class C.

Normally C is the class of all level α-tests.

Theorem 8.3.2. Neymann-Pearson lemma.

Consider testing $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$ using a test with rejection region R that satisfies

$\underline{x} \in R$, ~~then~~ if $\dfrac{L(\theta_1 | \underline{x})}{L(\theta_0 | \underline{x})} > k \iff f(\underline{x}|\theta_1) > k f(\underline{x}|\theta_0)$ (1)

and $\underline{x} \in R^c$ if $\dfrac{L(\theta_1 | \underline{x})}{L(\theta_0 | \underline{x})} < k \iff f(\underline{x}|\theta_1) < k f(\underline{x}|\theta_0)$ (2)

for some $k \geq 0$ and $P(\underline{X} \in R | \theta_0) = \alpha$.

Then A test that satisfies 8.3.1 and 8.3.2 is a UMP ~~test~~ level α- test.

Proof. Assume $f(\underline{x}|\theta_0)$ and $f(\underline{x}|\theta_1)$ are pdfs

Define a test function $\phi(\underline{x}) = \begin{cases} 1 & \text{if } x \in R \\ 0 & \text{if } x \in R^c \end{cases}$

and let $\phi'(\underline{x})$ be the test function of any other level α- test. Let $\beta(\theta)$ and $\beta'(\theta)$ be the corresponding power functions. $\phi'(\underline{x})$ is an indicator function and satisfies

$0 \leq \phi'(\underline{x}) \leq 1$

We have $$(\phi(\underline{x}) \overset{\substack{\geq 0 \\ \leq 0}}{-} \phi'(\underline{x}))(f(\underline{x}|\theta_1) \overset{\substack{> 0 \\ < 0}}{-} k f(\underline{x}|\theta_0)) \geq 0$$

Thereby $$0 \leq \int (\phi(\underline{x}) - \phi'(\underline{x}))(f(\underline{x}|\theta_1) - k f(\underline{x}|\theta_0))\, d\underline{x}$$

$$= \beta(\theta_1) - \beta'(\theta_1) - k\underbrace{(\beta(\theta_0) - \beta'(\theta_0))}_{\alpha \geq 0} \leq \beta(\theta_1) - \beta'(\theta_1)$$

$$\Rightarrow \beta(\theta_1) \geq \beta'(\theta_1)$$

8.3.13

If $T(\underline{x})$ is a sufficient statistic for θ with pdf/pmf $g(t|\theta)$, a test for $H_0: \theta = \theta_0$ versus $H_1: \theta = \theta_1$ based on T with rejection region $\underbrace{S}_{\text{Subset of sample space of } T.}$ is a UMP level α-test if it satisfies

$t \in S$ if $g(t|\theta_1) > k g(t|\theta_0)$

$t \in S^c$ if $g(t|\theta_1) < k g(t|\theta_0)$

for some $k \geq 0$ where $\alpha = P(T \in S | \theta_0)$

Proof. Let $f(\underline{x}|\theta_i) = g(T(\underline{x})|\theta_i) h(\underline{x})$ in (1) and (2)

Examples

$X \sim B(2,\theta)$ $\quad H_0: \theta = \frac{1}{2}$, $H_1: \theta = \frac{3}{4}$

Then $\frac{f(0|\frac{3}{4})}{f(0|\frac{1}{2})} = \frac{1}{4}$, $\frac{f(1|\frac{3}{4})}{f(1|\frac{1}{2})} = \frac{3}{4}$, $\frac{f(2|\frac{3}{4})}{f(2|\frac{1}{2})} = \frac{9}{4}$

For $\frac{3}{4} < k < \frac{9}{4}$ the UMP level-α test gives $R = \{2\}$ for X

For $\frac{1}{4} < k < \frac{3}{4}$ — " — " — $R = \{1,2\}$ for X —

Note $k = \frac{3}{4}$ leaves the test undecided for $X = 1$.

$X_1, \dots, X_n \sim N(\theta, \sigma^2)$

$H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$, ~~$\theta_0 \to \theta_1$~~ $\theta_0 > \theta_1$

$$g(\bar{x}|\theta_1) > k\, g(\bar{x}|\theta_0) \Rightarrow \frac{\frac{1}{\sqrt{2\pi}} \frac{\sqrt{n}}{\sigma} e^{-\frac{n}{2\sigma^2}(\bar{x}-\theta_1)^2}}{\frac{1}{\sqrt{2\pi}} \frac{\sqrt{n}}{\sigma} e^{-\frac{n}{2\sigma^2}(\bar{x}-\theta_0)^2}} > k$$

$$\Leftrightarrow (2\theta_1 - 2\theta_0)\bar{x} > \log k \frac{2\sigma^2}{n} + \theta_1^2 - \theta_0^2$$

$$\Rightarrow \bar{x} < \frac{\log k \frac{2\sigma^2}{n} + \theta_1^2 - \theta_0^2}{2(\theta_1 - \theta_0)} \quad \text{since } \theta_1 < \theta_0.$$

$P(\bar{X} \le c | \theta_0) = \alpha$ gives $R = \{x : \bar{x} < \theta_0 - \frac{\sigma z_\alpha}{\sqrt{n}}\}$.