2.3 Momentgenerating functions

$$M_X(t) \overset{\Delta}{=} E[e^{tX}] = \begin{cases} \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx, & X \text{ cont} \\ \sum_x e^{tx} P(X=x), & X \text{ discrete} \end{cases} \quad |t| < h.$$

$$M_X^{m}(t) = E[X^m e^{tX}]$$

$$M_X^{m}(0) = E[X^m]$$

Theorem 2.3.15

$Y = aX + b$, a constant

Then $M_Y(t) = M_{aX+b}(t) = e^{bt} M_X(at)$

Proof $E[e^{(aX+b)t}] = e^{bt} E[e^{atX}] = e^{bt} M_X(at)$

Example

$$X \sim N(\mu, \sigma^2), \quad f_X(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad \begin{matrix} -\infty < x < \infty \\ -\infty < \mu < \infty \\ \sigma^2 > 0 \end{matrix}$$

$$Z \sim N(0,1) \Rightarrow f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2}$$

$$M_Z(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} e^{tz}\,dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z^2-2tz+t^2)}{2}} e^{\frac{t^2}{2}}\,dz$$

$$= e^{\frac{t^2}{2}} \cdot \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z-t)^2}{2}}\,dz}_{1} = e^{\frac{t^2}{2}}$$

$$M_X(t) = M_{\sigma Z + \mu} = e^{\mu t} M_Z(\sigma t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$M_X'(t) = \left(e^{\mu t + \frac{\sigma^2 t^2}{2}}\right)(\mu + t\sigma^2), \quad M_X'(0) = \mu$$

$$M_X''(t) = \left(e^{\mu t + \frac{\sigma^2 t^2}{2}}\right)(\mu + t\sigma^2)^2 + \sigma^2 e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$M_X''(0) = \mu^2 + \sigma^2 \Rightarrow Var[X] = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$$

$X \sim$ lognormal if $\log X \sim N(\mu, \sigma^2)$

$Y = \log X \Rightarrow X = e^Y$, $Y \sim N(\mu, \sigma^2)$, $f_Y(y) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}}$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} \cdot \frac{1}{x} e^{-\frac{(\log x - \mu)^2}{2\sigma^2}}, \quad 0 < x < \infty$$

$$E[X] = E[e^Y] = M_Y(1) = e^{\mu + \frac{\sigma^2}{2}}$$

$$E[X^2] = E[e^{2Y}] = M_Y(2) = e^{2\mu + 2\sigma^2}$$

$$\Rightarrow Var[X] = e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2} = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1) \propto E[X]^2$$

Theorem 2.3.11

Let $F_X(x)$ and $F_Y(y)$ be two cdfs where all moments exist.

a) Suppose X and Y have bounded support. Then

$F_X(u) = F_Y(u)$, $\forall u \Leftrightarrow E[X^r] = E[Y^r]$, $r = 0, 1, 2, \ldots$

b) If $M_X(t)$ and $M_Y(t)$ exist and $M_X(t) = M_Y(t)$, $\forall |t| \le h$,

then $F_X(u) = F_Y(u)$, $\forall u$.

Theorem 2.3.12

Let $\{X_i, i = 1, 2, \ldots\}$ be a sequence of random variables, each with mgf $M_{X_i}(t)$

Suppose $\lim_{n \to \infty} M_{X_n}(t) = M_X(t)$ for $|t| \le h$

Then $\lim_{n \to \infty} F_{X_n}(x) = F_X(x)$

Example 2.3.13

$X \sim B(n,p) \Rightarrow M_X(t) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$, ~~$x=0,1,2,\dots$~~

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x} = (pe^t + 1-p)^n = \left(1 + \frac{1}{n}(e^t-1)np\right)^n$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Let $X_n \sim B(n, p_n)$

$$M_{X_n} = \left(1 + \frac{1}{n}(e^t-1)np_n\right)^n \overset{np_n=\lambda}{=} \left(1 + \frac{1}{n}(e^t-1)\lambda\right)^n \xrightarrow[\substack{n\to\infty \\ np_n=\lambda}]{} e^{\lambda(e^t-1)}$$

$Y \sim$ Poisson (λ)

$$M_Y(t) = \sum_{y=0}^{\infty} e^{ty} \frac{\lambda^y e^{-\lambda}}{y!} = e^{-\lambda} \sum_{y=0}^{\infty} \frac{(\lambda e^t)^y}{y!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

$$\Rightarrow F_{X_n}(x) \xrightarrow[\substack{n\to\infty \\ np_n=\lambda}]{} \text{Poisson}(\lambda)$$

Important result

$X_1, \dots, X_n$ independent with moment generating function $M_{X_i}(t) \Rightarrow$

$$M_{\Sigma X_i}(t) = E\left[e^{\sum_{i=1}^{n} X_i t}\right] = E\left[e^{X_1 t} \cdot e^{X_2 t} \cdot \ldots \cdot e^{X_n t}\right] = \prod_{i=1}^{n} M_{X_i}(t)$$

Differentiation under an Integral sign

Theorem 2.41 Leibnitz rule.

Let $f(x,\theta)$, $a(\theta)$ and $b(\theta)$ be differentiable with respect to θ, Then:

$$\frac{d}{d\theta}\int_{a(\theta)}^{b(\theta)} f(x,\theta)\,dx = f(b(\theta),\theta)\frac{d}{d\theta}b(\theta) - f(a(\theta),\theta)\frac{d}{d\theta}a(\theta) + \int_{a(\theta)}^{b(\theta)}\frac{\partial}{\partial\theta}f(x,\theta)\,dx$$

Chapter 3. Distributions

Bernoulli distribution

$X \sim$ Bernoulli(p) if

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \quad -\text{"}- \quad 1-p \end{cases}$$

$E[X] = p$, $E[X^2] = p$, $Var[X] = p(1-p)$

$$M_X(t) = \sum_{x=0}^{1} e^{tx} \cdot P(X=x) = (1-p+pe^t)$$

Sequence of independent trials

Reg A or A^c

$P(A) = p$

$X =$ the number of times A happens in n independent trials

$X \sim B(n,p)$. $X = \sum_{i=1}^{n} X_i$ where $X_i \sim$ Bernoulli(p)

and X_i, $i = 1, \dots, n$ are independent.

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \dots, n$$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = np, \quad Var[X] = Var\left[\sum_{i=1}^{n} X_i\right] = np(1-p)$$

$$X_1, \dots, X_n \text{ independ} \Rightarrow M_{\sum_{i=1}^{n} X_i}(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

$$\Rightarrow M_X(t) = (1-p+pe^t)^n$$

Negative binomial distribution

Let X be the number of trials until the r-th success

$P(X = x \mid r, p) =$ P(A occurs $r-1$ times in $x-1$ trials and A occurs in the x-th trial

$$= \binom{x-1}{r-1} p^{r-1} (1-p)^{x-r} p, \quad x = r, r+1, \ldots$$

$$= \binom{x-1}{r-1} p^{r} (1-p)^{x-r}, \quad x = r, r+1, \ldots$$

$r = 1$ gives

$$P(X = x \mid 1, p) = p(1-p)^{x-1}, \quad x = 1, 2, \ldots$$

which is the geometric distribution, for which

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} p (1-p)^{x-1} \overset{s = x-1}{=} pe^t \sum_{s=0}^{\infty} (e^t(1-p))^s = \frac{pe^t}{1-e^t(1-p)}$$

Need $e^t(1-p) < 1 \iff e^t < \frac{1}{1-p}$ or $t < \ln\left(\frac{1}{1-p}\right)$

$$M_X'(t) = \frac{pe^t}{1-e^t(1-p)} + \frac{pe^t\, e^t(1-p)}{(1-e^t(1-p))^2} = \frac{pe^t}{1-e^t(1-p)}\left[1 + \frac{e^t(1-p)}{1-e^t(1-p)}\right]$$

$$= \frac{pe^t}{(1-e^t(1-p))^2}. \quad M_X'(0) = \frac{1}{p} = E[X]$$

Similarly: $Var[X] = E[X^2] - (E[X])^2 = \frac{1-p}{p^2}.$