TMA4295 Statistical inference
Exercise 1 - solution

2.33

$M_X(t) = \mathrm{E}\left(e^{tX}\right)$, $\mathrm{E}\left(X^n\right) = \left.\frac{d^n M_X(t)}{dt^n}\right|_{t=0}$

a) Use the fact that $e^y = \sum\limits_{k=0}^{\infty} \frac{y^k}{k!}$ for the computation of the moment generating function.
$\mathrm{E}(X) = \lambda$
$\mathrm{E}\left(X^2\right) = \lambda^2 + \lambda$
$\mathrm{Var}(X) = \lambda$

c) Use completing the square

$$\begin{aligned} x^2 - 2\mu x - 2\sigma^2 t x + \mu^2 &= x^2 - 2(\mu + \sigma^2 t)x \pm (\mu + \sigma^2 t)^2 + \mu^2 \\ &= (x - (\mu + \sigma^2 t))^2 - (2\mu\sigma^2 t + (\sigma^2 t)^2) \end{aligned}$$

and the fact that integrals of the probability density functions over the probability space are equal to 1 (in this case it leads to the normal distribution) in the computation of the moment generating function.
$\mathrm{E}(X) = \mu$
$\mathrm{E}\left(X^2\right) = \mu^2 + \sigma^2$
$\mathrm{Var}(X) = \sigma^2$

2.35

a) Use the fact that $x^r = e^{r\log(x)}$ and the substitution $y = \log(x)$ and completing the square together with the form of the normal distribution as in the exercise 2.33c).

b) Use the same transformation $x^r = e^{r\log(x)}$ and substitution $y = \log(x) - r$. The resulting integral is an odd function so the negative integral cancels the positive one.

2.38

a) Use the fact that $\sum\limits_{x=0}^{\infty} \binom{r+x-1}{x} \left((1-p)e^t\right)^x \left(1 - (1-p)e^t\right)^r = 1$ for $(1-p)e^t < 1$, since this is just sum of the pmf of the negative binomial distribution.
$\mathrm{E}\left(e^{tX}\right) = \left(\frac{p}{1-(1-p)e^t}\right)^r$, $t < -\log(1-p)$

b) Use the fact, that $M_{2pX}(t) = M_X(2pt)$. The limit can be computed with use of the L'Hospital rule and the limiting moment generating function is the moment generating function of the χ^2 squared distribution with $2r$ degrees of freedom (see tables).

Problem 3.20
X random variable with the pdf $f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}$

a) Mean:

$$E(x) = \int_0^\infty \frac{2}{\sqrt{2\pi}} e^{-x^2/2} x dx = -\frac{2}{\sqrt{2\pi}} e^{-x^2/2} \Big|_0^\infty = \frac{2}{\sqrt{2\pi}}.$$

Variance: since $Var(X) = E(x^2) - E(x)^2$ we need to compute $E(x^2)$.

$$E(x^2) = \int_0^\infty \frac{2}{\sqrt{2\pi}} e^{-x^2/2} x^2 dx = -\frac{2}{\sqrt{2\pi}} e^{-x^2/2} x \Big|_0^\infty + \int_0^\infty \frac{2}{\sqrt{2\pi}} e^{-x^2/2} dx = 1$$

$$\Rightarrow Var(x) = 1 - \frac{2}{\pi}.$$

b) We notice using the transformation $y = x^2$ and so $x = \sqrt{y}$ that:

$$f_Y(y) = \frac{2}{\sqrt{2\pi}} \frac{1}{2\sqrt{y}} e^{-y/2} = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{2}} y^{-\frac{1}{2}} e^{-y/2} = \frac{1}{\Gamma(\frac{1}{2}) 2^{\frac{1}{2}}} y^{\frac{1}{2}-1} e^{(-y/2)}$$

that is gamma distributed with $\alpha = \frac{1}{2}$ and $\beta = 2$.

Problem 3.23
The Pareto distribution has pdf :

$$f(x) = \frac{\beta \alpha^\beta}{x^{\beta+1}} \quad \alpha < x < \beta, \quad \alpha > 0, \quad \beta > 0.$$

a) Verify that $f(x)$ is a pdf:

$$\int_\alpha^\infty \frac{\beta \alpha^\beta}{x^{\beta+1}} dx = \beta \alpha^\beta \left[-\frac{1}{\beta} x^{-\beta} \right]_\alpha^\infty = 1.$$

b) Mean and variance:

$$E(x) = \int_\alpha^\infty \frac{\beta \alpha^\beta}{x^{\beta+1}} x dx = \frac{\beta \alpha^\beta}{(\beta - 1)\alpha^{\beta-1}} = \frac{\beta \alpha}{\beta - 1}$$

$$E(x^2) = \int_\alpha^\infty \frac{\beta \alpha^\beta}{x^{\beta+1}} x^2 dx = \frac{\beta \alpha^\beta}{(\beta - 2)\alpha^{\beta-2}} = \frac{\beta \alpha^2}{\beta - 2}$$

$$\Rightarrow Var(x) = \frac{\beta \alpha^2}{\beta - 2} - \left(\frac{\beta \alpha}{\beta - 1} \right)^2.$$

c) $E(x^2)$ does not exist for $\beta < 2 \Rightarrow$ the variance does not exist.