TMA4295 Statistical inference
Exercise 11 - solution

Problem 1
Using the factorization theorem

$$f(\mathbf{X}|\theta) = \theta^{n\theta}\left[\Gamma(\theta)\right]^{-m}(X_1\cdot\dots\cdot X_n)^{\theta-1}e^{-\theta\sum_i X_i} \qquad \text{with } \theta > 0$$

$$= \theta^{n\theta}\left[\Gamma(\theta)\right]^{-m}(X_1\cdot\dots\cdot X_n e^{-\sum_i X_i})^{\theta}(X_1\cdot\dots\cdot X_n)^{-1}.$$

Put

$$T(X_i \dots X_n) = \prod X_i \cdot e^{-\sum X_i}$$

$$g(T,\theta) = \theta^{n\theta}\left[\Gamma(\theta)\right]^{-m}\left[T(X_1\dots X_n)\right]^{\theta}$$

$$h(X_1,\dots,X_n) = (X_1\cdot\dots\cdot X_n)^{-1}$$

Problem 2

a) $Y = \sum_{i=1}^n X_i$, X_1, $i = 1,\dots,n$ are i.i.d. $B(1,p)$

$$\Rightarrow B(n,p) \Rightarrow f(y|p) = \binom{n}{y}p^y(1-p)^{n-y}$$

$$= \binom{n}{y}(\frac{p}{1-p})^y(1-p)^n = \binom{n}{y}exp\left(y\log\left(\frac{p}{1-p}\right)\right)(1-p)^n$$

With $h(y) = \binom{n}{y}$, $c(p) = (1-p)^n$, $w(p) = \log\left(\frac{p}{1-p}\right)$, $t = y$ we get that is an exponential family.

$$f(y|p) = \binom{n}{y}p^y(1-p)^{n-y} = h(y)g(y|p)$$

Shows that Y is a sufficient statistics for p.

b) For two samples $x_1,\dots,x_n$ and $z_1,\dots,z_n$, $y = \sum x_i$ and $z = \sum z_i$ we have $\frac{\binom{n}{y}p^y(1-p)^{n-y}}{\binom{n}{z}p^z(1-p)^{n-z}}$ constant as a function of p $\iff$ $y = z \Rightarrow y = \sum x_i$ is minimal sufficient.
Y is from an exponential family with $p \in (0,1)$, hence Y is a complete statistics.

c) $$P(X_1 = 1|Y = y) = \frac{P\left(X_1 = 1 \cap \sum_{i=2}^n X_i = y-1\right)}{P(Y=y)} = \frac{p\binom{n-1}{y-1}p^{y-1}(1-p)^{n-y}}{\binom{n}{y}p^y(1-p)^{n-y}} = \frac{y}{n}$$

$$= \frac{\sum x_i}{n} \Rightarrow E(X_1|Y) = \frac{\sum x_i}{n} \quad \text{and} \quad E\left(\frac{\sum x_i}{n}\right) = p$$

and it is the unique best unbiased estimator of p.

Problem 3

a) Denote the Fisher information of the sample and that of one observation by $I(\theta)$ and $I_0(\theta)$ respectively. Then $I(\theta) = nI_0(\theta)$ and

$$I_0(\theta) = E\left(\frac{\partial \ln f(X;\theta)}{\partial \theta}\right)^2 = E\left(\frac{X^2}{\theta^3} - \frac{1}{\theta}\right)^2 = \frac{1}{\theta^6}EX^4 - \frac{1}{\theta^2}$$

To find EX^4 we can use mgf: $EX^n = M_X{}^{(n)}(0)$.

We have

$$M_X(t) = e^{\theta^2 t^2/2}$$

$$M_X''(t) = e^{\theta^2 t^2/2}(\theta^4 t^2 + \theta^2)$$

$$M_X'''(t) = e^{\theta^2 t^2/2}(\theta^6 t^3 + 3\theta^4 t)$$

$$M_X^{(4)}(t) = e^{\theta^2 t^2/2}(\theta^2 t)(\theta^6 t^3 + 3\theta^4 t) + e^{\theta^2 t^2/2}(3\theta^6 t^2 + 3\theta^4)$$

Hence $EX^4 = M^{(4)}(0) = 3\theta^4$

and

$$I_0(\theta) = \frac{2}{\theta^2}, \quad I(\theta) = \frac{2n}{\theta^2}$$

b)

$$ET_n = \frac{2}{n}EX_1^2 + \frac{(n-2)}{n(n-1)}\sum_{i=2}^{n} EX_i^2 = \theta^2\left(\frac{2}{n} + \frac{(n-2)}{n(n-1)}(n-1)\right) = \theta^2$$

i.e. T_n is unbiased.

c)

$$VarT_n = \frac{4}{n^2}Var(X_1^2) + \frac{(n-2)^2}{n^2(n-1)^2}\sum_{i=2}^{n} Var(X_i^2) =$$

$$= Var(X^2)\left[\frac{4}{n^2} + \frac{(n-2)^2(n-1)}{n^2(n-1)^2}\right] = Var(X^2)\frac{1}{n-1} = \frac{2\theta^4}{n-1}$$

since

$$Var(X^2) = EX^4 - (EX^2)^2 = 3\theta^4 - \theta^4 = 2\theta^4$$

(EX^4 was obtained in part (a)).

The Cramer-Rao lower bound is (use part (a))

$$\frac{\left[\frac{d}{d\theta}(\theta^2)\right]^2}{I(\theta)} = \frac{4\theta^2}{2n/\theta^2} = \frac{2\theta^4}{n} < \frac{2\theta^4}{n-1} = Var(T_n).$$

Hence T_n is not efficient

d)

$$L(\theta; X_1, ..., X_n) = (2\pi)^{-n/2}\theta^{-n}e^{-\frac{1}{2\theta^2}\sum X_i^2}.$$

$$\frac{\partial \ln L}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{\theta^3}\sum X_i^2.$$

Problem 4

$X \sim Poisson(\alpha)$ and $Y \sim Poisson(\beta)$

a) $L(\alpha, \beta | x, y) = \frac{\alpha^x}{x!} e^{-\alpha} \frac{\beta^y}{y!} e^{-\beta}$. The MLE for α and β in the full model can be computed by taking the first derivatives respect to α and β of the log-likelihood and set the derivatives equal to zero.

$$\hat{\alpha} = x,$$

$$\hat{\beta} = y.$$

The MLE for α and β under H_0 can be computed as we did previously, keeping in mind that under H_0 we have $\alpha = \beta$.

$$\hat{\alpha}_0 = \frac{x+y}{2},$$

$$\hat{\beta}_0 = \frac{x+y}{2}.$$

b)

$$\lambda(x, y) = \frac{L(\hat{\alpha}_0, \hat{\beta}_0 | x, y)}{L(\hat{\alpha}, \hat{\beta} | x, y)} = \frac{\frac{\left(\frac{x+y}{2}\right)^x}{x!} e^{-\left(\frac{x+y}{2}\right)} \frac{\left(\frac{x+y}{2}\right)^y}{y!} e^{-\left(\frac{x+y}{2}\right)}}{\frac{x^x}{x!} e^{-x} \frac{y^y}{y!} e^{-y}} = \left(\frac{x+y}{2x}\right)^x \left(\frac{x+y}{2x}\right)^y.$$