TMA4295 Statistical inference
Exercise 12 - solution

Problem 1

a) $X_1, ..., X_n$ i.i.d $f(x-\mu)$ and we want to prove that $\bar{X}-\mu$ is a pivot.

Consider $X_i = (X_i - \mu) + \mu = Z_i + \mu$, then we have $\bar{X} - \mu = \bar{Z} + \mu - \mu = \bar{Z}$. It is easy to verify that the distribution of Z_i does not depend on μ, and so also $\bar{Z}$. Hence $\bar{X} - \mu$ is a pivot.

b) $X_1, ..., X_n$ i.i.d $\frac{1}{\sigma} f\left(\frac{x}{\sigma}\right)$ and we want to prove that $\frac{\bar{X}}{\sigma}$ is a pivot.

Consider $X_i = \sigma \frac{X_i}{\sigma} = \sigma Z_i$, then we have $\frac{\bar{X}}{\sigma} = \sigma \frac{\bar{Z}}{\sigma} = \bar{Z}$. It is easy to verify that the distribution of Z_i does not depend on σ, and so also $\bar{Z}$. Hence $\frac{\bar{X}}{\sigma}$ is a pivot.

c) $X_1, ..., X_n$ i.i.d $\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$ and we want to prove that $\frac{\bar{X}-\mu}{S}$ is a pivot.

consider $X_i = (X_i - \mu) + \mu = \sigma \frac{(X_i - \mu)}{\sigma} + \mu = \sigma Z_i + \mu$ then $\bar{X} - \mu = \sigma \bar{Z}$ and $S^2 = \frac{1}{1-n} \sum_{i=1}^{n} (\sigma Z_i - \sigma \bar{Z})^2 = \sigma^2 S_Z^2$. This implies that $\frac{\bar{X}-\mu}{S} = \frac{\sigma \bar{Z}}{\sigma S_Z}$. It is easy to verify that the distribution of Z_i does not depend on μ and σ, hence also $\bar{Z}$. Then $\frac{\bar{X}-\mu}{S}$ is a pivot.

Problem 2

Using theorem 2.1.10 we have that $F_T(T|\theta)$ is uniform(0,1) and is a pivot. Hence we have

$$P_{\theta_0}\left(\{T : \alpha_1 \leq F_T(T|\theta_0) \leq 1 - \alpha_2\}\right) = P(\alpha_1 \leq U \leq 1 - \alpha_2) = 1 - \alpha_1 - \alpha_2 = 1 - \alpha$$

where $U \sim uniform(0,1)$. Then we have as an α level acceptance region

$$\{t : \alpha_1 \leq \ F_T(t|\theta_0) \leq 1 - \alpha_2\}$$

and as $\alpha - 1$ confidence interval

$$\{\theta : \alpha_1 \leq \ F_T(t|\theta) \leq 1 - \alpha_2\}.$$

Problem 3

The confidence interval derived by the method of Section 9.2.3 is

$$C(y) = \left\{\mu \colon y + \frac{1}{n}\log\left(\frac{\alpha}{2}\right) \le \mu \le y + \frac{1}{n}\log\left(1 - \frac{\alpha}{2}\right)\right\}$$

where $y = \min_i x_i$. The LRT method derives its interval from the test of $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$. Since Y is sufficient for μ, we can use $f_Y(y \mid \mu)$. We have

$$\begin{aligned}
\lambda(y) &= \frac{\sup_{\mu=\mu_0} L(\mu|y)}{\sup_{\mu\in(-\infty,\infty)} L(\mu|y)} &=& \frac{ne^{-n}(y-\mu_0)I_{[\mu_0,\infty)(y)}}{ne^{-(y-y)}I_{[\mu,\infty)(y)}} \\
&= e^{-n(y-\mu_0)}I_{[\mu_0,\infty)}(y) &=& \begin{cases} 0 & \text{if } y < \mu_0 \\ e^{-n(y-\mu_0)} & \text{if } y \ge \mu_0. \end{cases}
\end{aligned}$$

We reject H_0 if $\lambda(y) = e^{-n(y-\mu_0)} < c_\alpha$, where $0 \le c_\alpha \le 1$ is chosen to give the test level α. To determine c_α, set

$$\begin{aligned}
\alpha &= P\{\text{reject } H_0 | \, \mu = \mu_0\} &=& P\left\{Y > \mu_0 - \frac{\log c_\alpha}{n} \text{ or } Y < \mu_0 \,\middle|\, \mu = \mu_0\right\} \\
&= P\left\{Y > \mu_0 - \frac{\log c_\alpha}{n} \,\middle|\, \mu = \mu_0\right\} &=& \int_{\mu_0 - \frac{\log c_\alpha}{n}}^{\infty} ne^{-n(y-\mu_0)}\,dy \\
&= -e^{-n(y-\mu_0)}\Big|_{\mu_0 - \frac{\log c_\alpha}{n}}^{\infty} \quad = \quad e^{\log c_\alpha} &=& c_\alpha.
\end{aligned}$$

Therefore, $c_\alpha = \alpha$ and the $1-\alpha$ confidence interval is

$$C(y) = \left\{\mu \colon \mu \le y \le \mu - \frac{\log \alpha}{n}\right\} = \left\{\mu \colon y + \frac{1}{n}\ \log \alpha \le \mu \le y\right\}.$$

To use the pivotal method, note that since μ is a location parameter, a natural pivotal quantity is $Z = Y - \mu$. Then, $f_Z(z) = ne^{-nz}I_{(0,\infty)}(z)$. Let $P\{a \le Z \le b\} = 1 - \alpha$, where a and b satisfy

$$\begin{aligned}
\frac{\alpha}{2} = \int_0^a ne^{-nz}\,dz = -e^{-nz}\big|_0^a = 1 - e^{-na} \quad &\Rightarrow \quad e^{-na} = 1 - \frac{\alpha}{2} \\
&\Rightarrow \quad a = \frac{-\log\left(1 - \frac{\alpha}{2}\right)}{n} \\
\frac{\alpha}{2} = \int_b^\infty ne^{-nz}\,dz = -e^{-nz}\big|_b^\infty = e^{-nb} \quad &\Rightarrow \quad -nb = \log\frac{\alpha}{2} \\
&\Rightarrow \quad b = -\frac{1}{n}\log\left(\frac{\alpha}{2}\right)
\end{aligned}$$

Thus, the pivotal interval is $Y + \log(\alpha/2)/n \le \mu \le Y + \log(1 - \alpha/2)$, the same interval as from Example 9.2.13. To compare the intervals we compare their lengths. We have

$$\begin{aligned}
\text{Length of LRT interval} &= y - (y + \frac{1}{n}\log\alpha) &=& -\frac{1}{n}\log\alpha \\
\text{Length of Pivotal interval} &= \left(y + \frac{1}{n}\log(1-\alpha/2)\right) - (y + \frac{1}{n}\log\alpha/2) &=& \frac{1}{n}\log\frac{1-\alpha/2}{\alpha/2}
\end{aligned}$$

Thus, the LRT interval is shorter if $-\log\alpha < \log[(1-\alpha/2)/(\alpha/2)]$, but this is always satisfied.

Problem 4

a) The acceptance region: $\lambda(X) \geq c$ where

$$\lambda(X) = \frac{L(\theta_0; X)}{\sup L(\theta; X)} = \frac{L(\theta_0; X)}{L(\hat{\theta}_{MLE}; X)},$$

and c is found from the condition

$$P_{\theta_0}(\lambda(X) \geq c) = 1 - \alpha$$

$$\hat{\theta}_{MLE} = \frac{1}{n} \sum X_i = \bar{X}$$

$$\lambda(X) = e^{-\frac{n}{2}(\bar{X} - \theta_0)^2}$$

So, the acceptance region

$$\theta_0 - \frac{1}{\sqrt{n}} l_{1-\frac{\alpha}{2}} \leq \bar{X} \leq \theta_0 + \frac{1}{\sqrt{n}} l_{1-\frac{\alpha}{2}}$$

where l_δ - δ-quantile of the standard normal distribution.

b) Inverting the test of part (a) we obtain the following $(1-\alpha)$ confidence interval:

$$\left[\bar{X} - \frac{1}{\sqrt{n}} l_{1-\frac{\alpha}{2}}, \bar{X} + \frac{1}{\sqrt{n}} l_{1-\frac{\alpha}{2}}\right].$$

Problem 5

$X_1, ..., X_n$ i.i.d. $N(\mu, \sigma^2)$, we want to test

$$H_0 : \sigma^2 = \sigma_0^2 \; vs \; H_1 : \sigma^2 \neq \sigma_0^2$$

a) $\lambda(x, y) = \frac{\sup_{\Theta_0} L(\theta|x)}{\sup_{\Theta} L(\theta|x)}$, so we need to find the the values of θ that maximize L under H_0 and for the general case.

Here the parameter is $\theta = (\mu, \theta)$, for the general case the values that maximize L are obtained with the maximum likelihood estimation

$$\hat{\mu} = \bar{X}$$

$$\hat{\sigma^2} = \frac{1}{n} \sum_{i=1}^{n} (X - \hat{\mu})^2.$$

While under H_0, the parameter space is $\Theta_0 = \{(\mu, \sigma_0^2) : \infty < \mu < \infty\}$. The likelihood function for the Gaussian distribution is given by $L(\mu, \sigma^2) = \left(\frac{1}{\sqrt{2\pi\sigma}}\right)^n \exp\left(-\sum_{i=1}^n \frac{(x_i - \mu)^2}{2\sigma^2}\right)$, and under H_0 (so with $\sigma = \sigma_0$) it is easy to verify that the maximum in L is reached with

$$\hat{\mu}_0 = \bar{X}.$$

Then

$$\lambda(x, y) = \frac{\left(\frac{1}{\sqrt{2\pi\sigma_0}}\right)^n \exp\left(-\sum_{i=1}^n \frac{(x_i - \bar{X})^2}{2\sigma_0^2}\right)}{\left(\frac{1}{\sqrt{2\pi \frac{1}{n} \sum_{i=1}^n (X - \hat{\mu})^2}}\right)^n \exp\left(-\sum_{i=1}^n \frac{(x_i - \bar{X})^2}{2 \frac{1}{n} \sum_{i=1}^n (X - \hat{\mu})^2}\right)} = \left(\frac{Z}{n}\right)^{n/2} \exp((n - Z)/2)$$

with $Z = \frac{n\hat{\sigma}^2}{\sigma_0^2}$.

b) Note that

$$-2\ln\lambda(\mathbf{X}) = -2\left(\frac{n}{2}\ln\left(\frac{Z}{n}\right) + \frac{n-Z}{2}\right) = Z - n - n\ln\frac{Z}{n} = g(Z).$$

Looking at the plot of $g(z)$ we observe that the rejection region is then given by $R = \{z : g(z) \geq -2\ln c\} = \{z : z - n - n\ln\frac{z}{n} \geq -2\ln c\} = \{Z \leq z_0 \text{ } or \text{ } Z \geq z_1\}$.

c) We can notice that with $z_0 = 3.18$ and $z_1 = 22.91$ we have $g(z_0) = g(z_1)$ and from the cumulative distribution function of a χ^2_9 we have that $P(Z \leq z_0) + P(Z \geq z_1) = 0.05$

d) Here the main difference is that I'm looking at $g(Z)$ instead of Z. Hence the rejection region is $R = \{z : g(z) > -2\ln c\} = \{z : g(z) > C\}$. Since $g(Z) \sim \chi^2_1$ and $\alpha = 0.05 = P(Z \in R) = P(g(Z) \geq C)$ we get $C = 3.84$. Then it will be enough to find z_0 and z_1 such that $g(z_0) = g(z_1) = 3.84$

e) Using the Central Limit Theorem we have

$$\frac{Z - (n-1)}{\sqrt{2(n-1)}} \to N(0,1) \quad \text{in distribution}$$

now we notice that

$$U_n = \frac{Z - n}{\sqrt{(2n)}} = \frac{\sqrt{2(n-1)}}{\sqrt{2n}}\frac{Z-(n-1)}{\sqrt{2(n-1)}} - \frac{1}{\sqrt{2n}} \to N(0,1) \quad \text{in distribution.}$$

Since

$$-2\ln\lambda(\mathbf{X}) = \sqrt{2n}U_n - n\ln\left(\frac{U_n\sqrt{2}}{\sqrt{n}} + 1\right) \approx \sqrt{2n}U_n - n\left(\frac{U_n\sqrt{2}}{\sqrt{n}} - \frac{1}{2}\frac{2U_n^2}{n}\right) = U_n^2$$

and $U_n \sim \chi^2_1$ we have that $-2\ln\lambda(\mathbf{X}) \sim \chi^2_1$.